

# NOVEL GUIDANCE AND NONLINEAR CONTROL SOLUTIONS IN PRECISION GUIDED PROJECTILES

RICHARD A. HULL  
TECHNICAL FELLOW  
COLLINS AEROSPACE

JUNE 30, 2020



# OUTLINE:

## I. Introduction to Precision Guided Projectiles

- SGP 5” Projectile Flight Test Video

- Guidance, Navigation, Control Block Diagram

- Guidance, Navigation and Control Issues

## II. Guidance Law – GENEX

## III. Nonlinear Robust Control Law Design

- Background – Dynamic Inversion, Feedback Linearization, Back-stepping, Robust Recursive

## IV. Nonlinear Recursive Pitch-Yaw Autopilot Design for a Guided Projectile

- Nonlinear Recursive Pitch-Yaw Controller Design

- Aerodynamic Functions for Hypothetical Projectile

- Pitch-Yaw Controller Equations

- Mass Properties for Hypothetical Projectile

- MATLAB Simulation Results for Hypothetical Pitch-Yaw Projectile Controller

## V. Conclusions

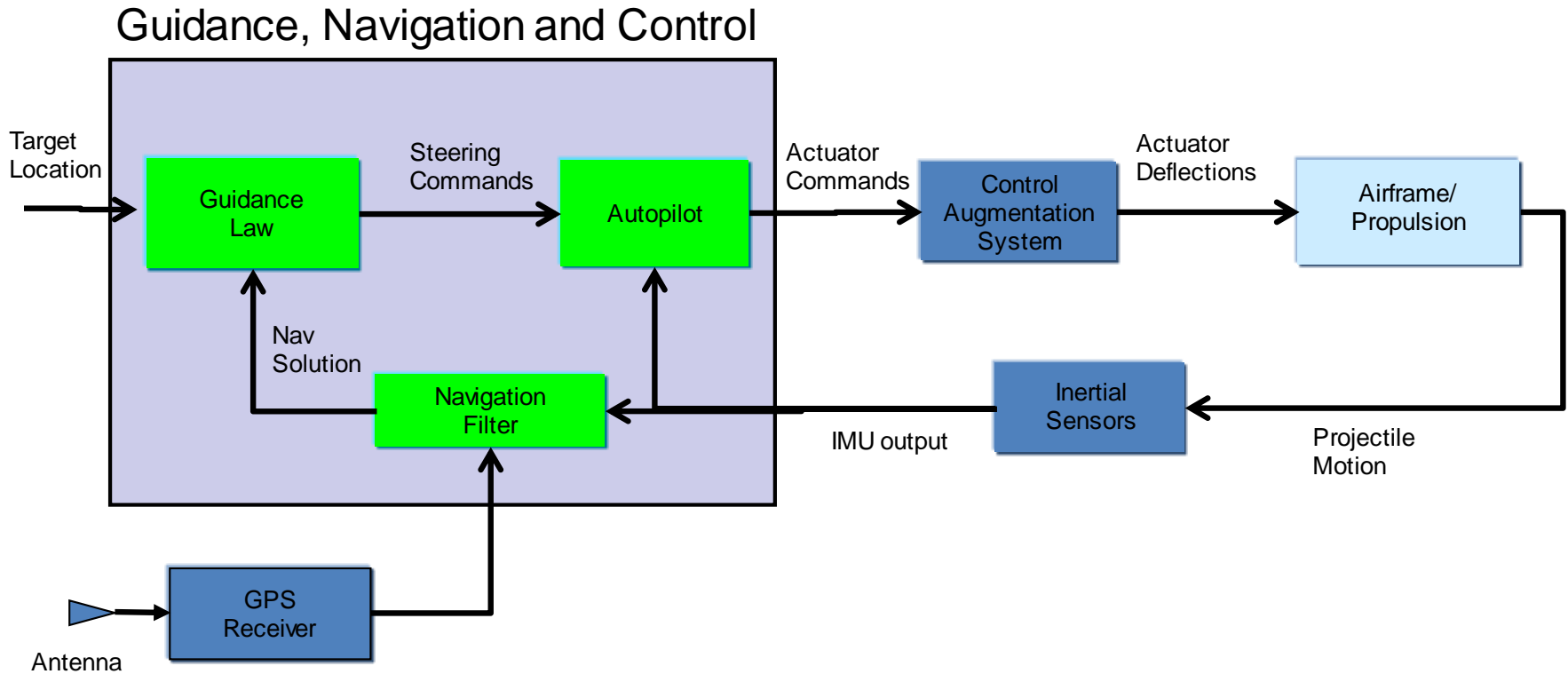
# 5-inch Multi Service Standard Guided Projectile

- Video Courtesy of BAE Systems



<https://www.bing.com/videos/search?q=Navy+5+inch+guided+projectile+video&view=detail&mid=E67446D3A21E0F0E9D7FE67446D3A21E0F0E9D7F&FORM=VIRE>

# SYSTEM BLOCK DIAGRAM



# GUIDANCE ISSUES:

Minimum Control Effort Optimal Control Problem with:

- Final Position (Miss Distance) Constraint
- Terminal Angle of Fall (Velocity Unit Vector) Constraint
- Control Action Constraint (Normal to Velocity Vector)
- Nonlinear Dynamics (True Optimal Solution is TPBV Problem)
- May be Final Time (Time on Target) Constraint

Several Well Known Sub-Optimal Solutions

- Modifications to Biased Proportional Nav
- Explicit Guidance
- Generalized Explicit Guidance (GENEX)
- Not well suited to Guided Projectiles of “Limited Maneuver Capability”

# NAVIGATION ISSUES:

- Electronics Must Be Densely Packaged and Low Cost!
- Inertial Sensors Must Survive High “G” Gun Launch Environment (~10,000 g’s)
- GPS Must Acquire Quickly, In-Flight, On A Rapidly Moving, Spinning Projectile
- Navigation Filter Must Initialize and Self-Orient In Mid-Flight
- What to Do if GPS is Unavailable

# CONTROL ISSUES:

- Ballistic Portion of Flight Must Avoid Roll Resonance Issues
- Airframe Control Capture at Canard Deployment
  - Airframe May Become Unstable at Canard Deploy
  - IMU Sensors May Be Saturated at Canard Deploy
    - High Roll Rate
    - Deployment Shock
- Large Flight Envelope of Mach and Dynamic Pressure
- Rapidly Changing Flight Envelope
- Winds and Variations in Atmospheric Properties Cause Significant Uncertainties in Gain Scheduled Parameters (No Air Data Probes)
- Highly Nonlinear and Uncertain Aerodynamics
  - Canard “Vortex Shedding” Interactions with Tails
  - Uncertain Tail “Clocking” Relative to Canards

# EXPLICIT GUIDANCE

**Solution to a Time Varying Linear Optimal Control Problem of the form:**

$$\text{minimize } J = \int_0^{t_f} \frac{u^2}{2} dt$$

**Subject to the following state constraints:**

$$\dot{x}_1 = x_2 - Tu$$

$$\dot{x}_2 = -u$$

**Which yields the optimal control:**

$$u = \frac{1}{T^2} [6x_1 - 2Tx_2]$$

**Where we define:**

$x_1 = y_f - y - \dot{y}T =$  position error

$x_2 = \dot{y}_f - \dot{y} =$  velocity error

$T = t_f - t =$  time - to - go

$u =$  acceleration control

Note – we often augment this with an acceleration term to include known effects of gravity and drag.



# GENERALIZED EXPLICIT GUIDANCE (GENEX)

“Generalized Vector Explicit Guidance”, Ernest J. Ohlmeyer and Craig A. Phillips, *AIAA Journal of Guidance, Control and Dynamics*, Vol. 29, No. 2, March-April 2006

## Solution to a Time Varying Linear Optimal Control Problem of the form:

$$\text{minimize } J = \int_0^{t_f} \frac{u^2}{2T^n} dt$$

Family of functions, parameterized by scalar  $n$   
Higher  $n$  allows greater penalty weight on control usage as  $T \rightarrow 0$

## Subject to the following state constraints:

$$\dot{x}_1 = x_2 - Tu$$

$$\dot{x}_2 = -u$$

## Where we define:

$$x_1 = y_f - y - \dot{y}T = \text{position error}$$

$$x_2 = \dot{y}_f - \dot{y} = \text{velocity error}$$

$$T = t_f - t = \text{time-to-go}$$

$$u = \text{acceleration control}$$

## Which yields the optimal control:

$$u = \frac{1}{T^2} [k_1 x_1 + k_2 x_2 T]$$

where :

$$k_1 = (n+2)(n+3)$$

$$k_2 = -(n+1)(n+2)$$

Note that  $n = 0$  results in  $k_1 = 6$  and  $k_2 = -2$  reducing to the standard explicit guidance gains

# GENEX with Gravity Term (NEW):

Note on Updated Derivation of Generalize Explicit Guidance to Include Gravity Acceleration Term from Ernie Ohlmeyer, May 2016

## Solution to a Time Varying Linear Optimal Control Problem of the form:

$$\text{minimize } J = \int_0^{t_f} \frac{u^2}{2T^n} dt$$

Family of functions, parameterized by scalar  $n$   
Higher  $n$  allows greater penalty weight on control usage as  $T \rightarrow 0$

## Subject to the following state constraints:

$$\dot{x}_1 = x_2 - Tu$$

$$\dot{x}_2 = -u$$

## Which yields the optimal control:

$$u = \frac{1}{T^2} [k_1 x_1 + k_2 x_2 T] + k_3 g$$

where:

$$k_1 = (n+2)(n+3)$$
$$k_2 = -(n+1)(n+2)$$
$$k_3 = \frac{(n+2)(n-1)}{2}$$

## Where we define:

$$x_1 = y_f - y - \dot{y}T - \frac{1}{2}gT^2 = \text{position error}$$

$$x_2 = \dot{y}_f - \dot{y} - gT = \text{velocity error}$$

$$T = t_f - t = \text{time-to-go}$$

$$u = \text{acceleration control}$$

$$g = \text{acceleration of gravity}$$

Setting  $k_3 = 0$  recovers original GENEX

# EXAMPLE TRAJECTORIES USING GENEX

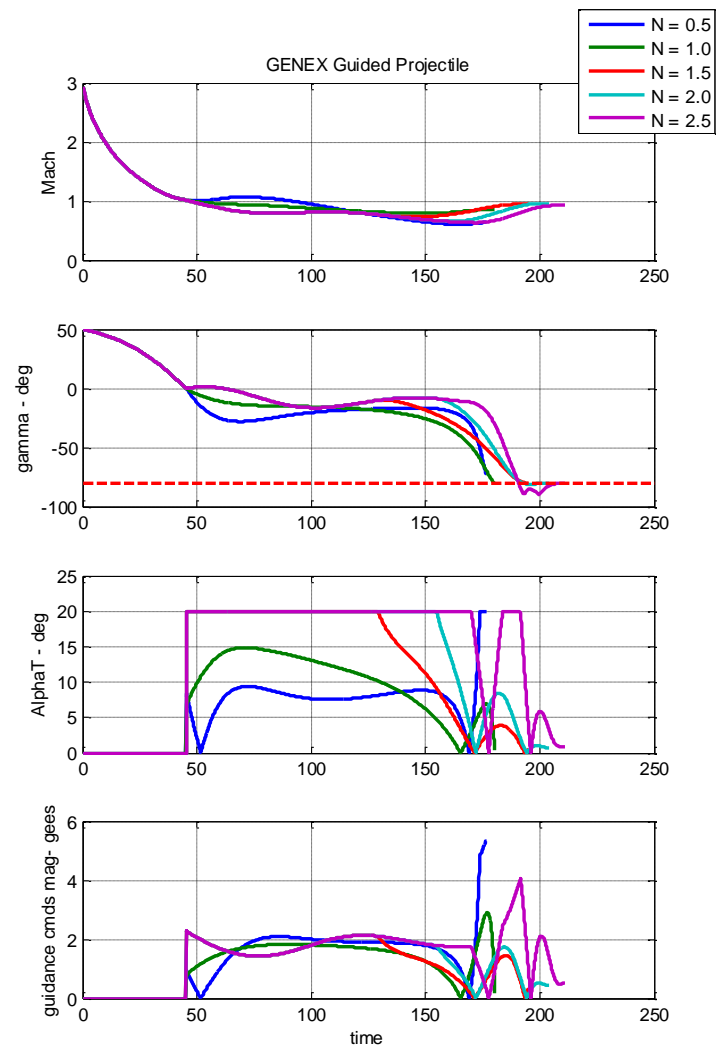
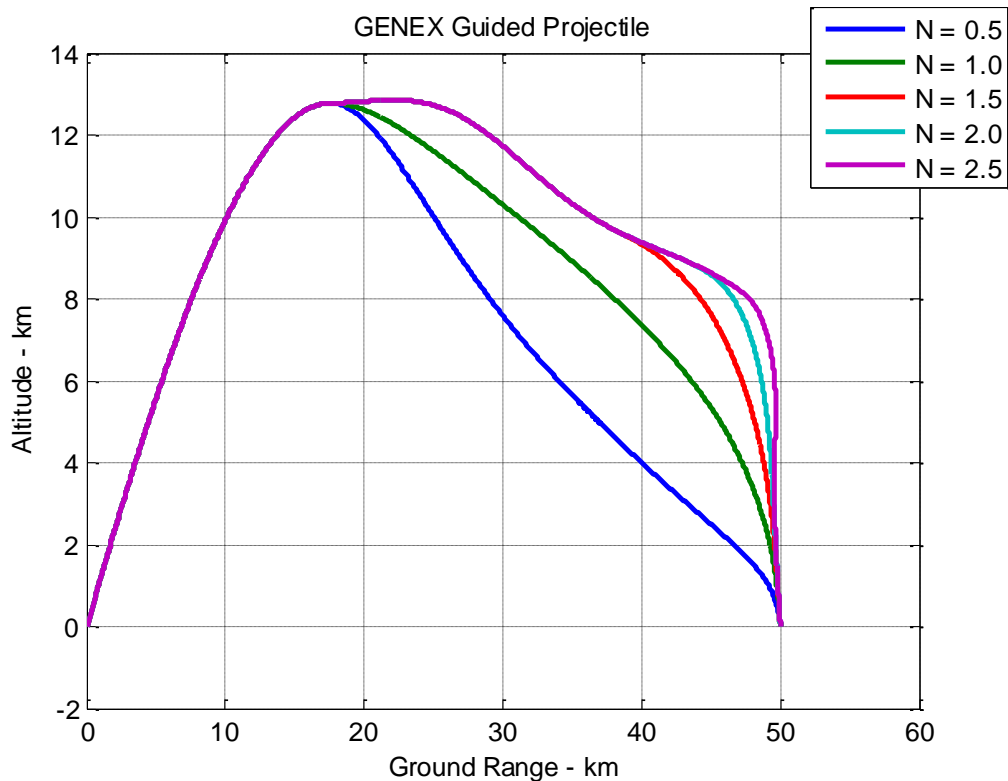
GENEX Guidance from Apogee

$V_0 = 1000$  m/s

Launch Elevation = 50 deg

Final Gamma = -80 deg

Target Range = 50 km

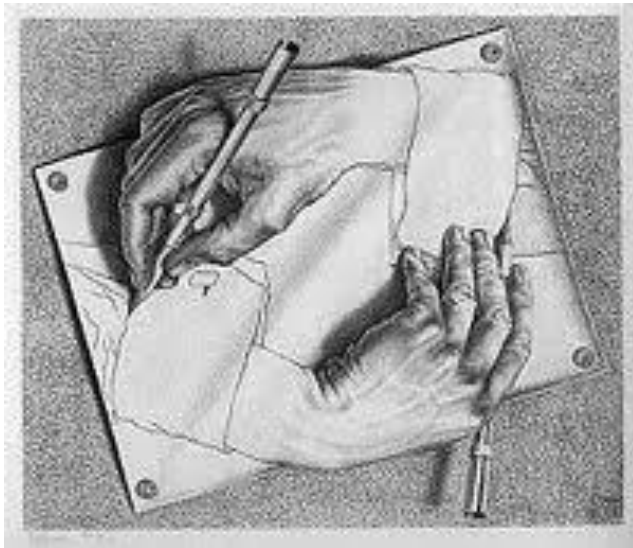


# RECURSIVE CONTROL DESIGN:



*“Recursivist”* – one who views the world as a giant opportunity to apply the rigorous methods of *recursive* nonlinear control design, one layer at a time!

See also *“Integrator Backsteppinger”*, *“Dynamic Inversionist”* and *“Feedback Linearizationist”*



# DYNAMIC INVERSION

Given a nonlinear system “affine” w.r.t. the control input:

$$\dot{x} = f(x) + g(x)u$$

$$x \in \mathfrak{R}^n, \quad u \in \mathfrak{R}^n$$

If  $g(x)$  is invertible the control:

$$u = g^{-1}(x)[\dot{x}_d - f(x)]$$

Will cause the system to track the desired dynamics :

$$\dot{x} = \dot{x}_d$$

Note that we are not actually inverting the dynamics of the entire system, only the algebraic input connection matrix!

# DYNAMIC INVERSION

The primary restrictions on Dynamic Inversion are:

1. The system must be scalar or *square* – that is it must have the same number of inputs as states, and
2. The matrix  $g(x)$  must be non-singular *over the entire region of interest*.

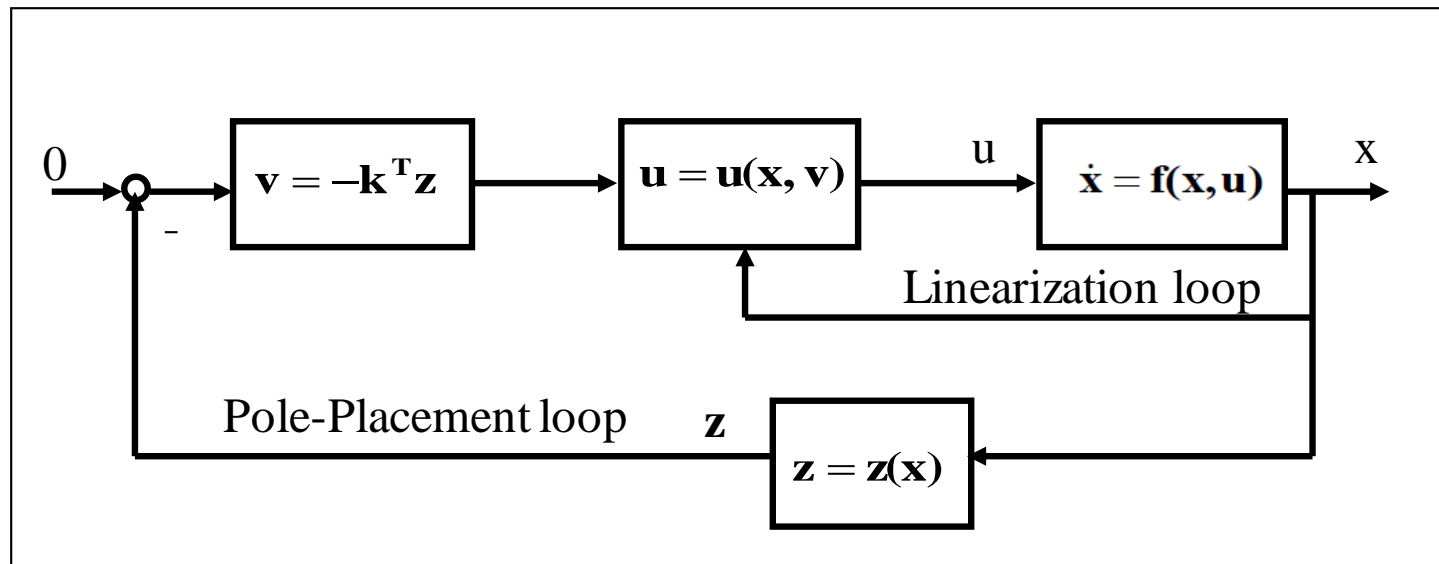
Invertibility of  $g(x)$  also insures stability of the zero dynamics, so that the system is minimum phase.

In fact, it is a stronger condition, which guarantees that the system can be decoupled into  $n$  independently controllable subsystems by the  $n$  control inputs.

This condition is somewhat rare in real systems!

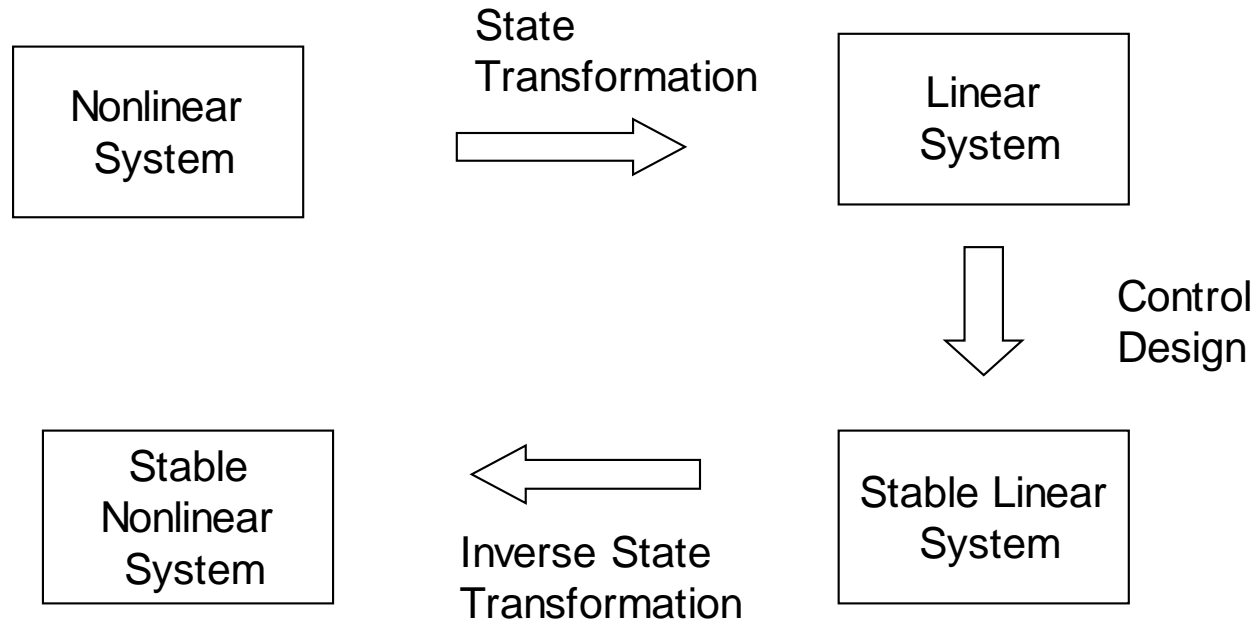
# FEEDBACK LINEARIZATION

- Not Conventional (Jacobi) Linearization
- Systematic Method for finding a state transformation that maps the system to an “equivalent” linear system.
- Design Control for the Linear System
- Map the Control back to the original nonlinear system.



# FEEDBACK LINEARIZATION

Feedback Linearization involves a transformation of state variables in order to make the control design and stability analysis of the system easier.



But, in order to guarantee “equivalence” of dynamics and transference of stability properties between the linear and nonlinear systems, we must rely on concepts from set theory, and differential geometry.



# FEEDBACK LINEARIZATION - COMMENTS

Input-Output Feedback Linearization is more desirable for the control of missiles and aircraft, in which output tracking (command following) is of interest.

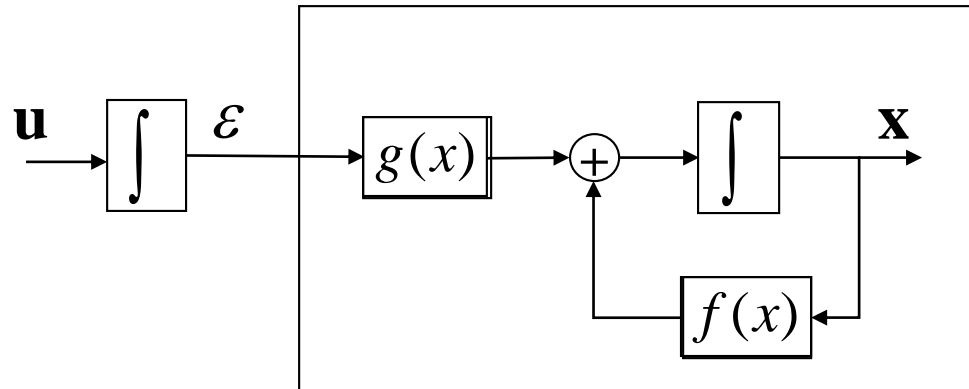
Exact Output Tracking is not possible for non-minimum phase systems, which are equivalent to systems with less than full relative degree ( $r < n$ ), and unstable zero dynamics.

Tail Control of Aircraft and Missiles exhibits minimum phase response in alpha and beta, but non-minimum phase characteristics in normal accelerations.

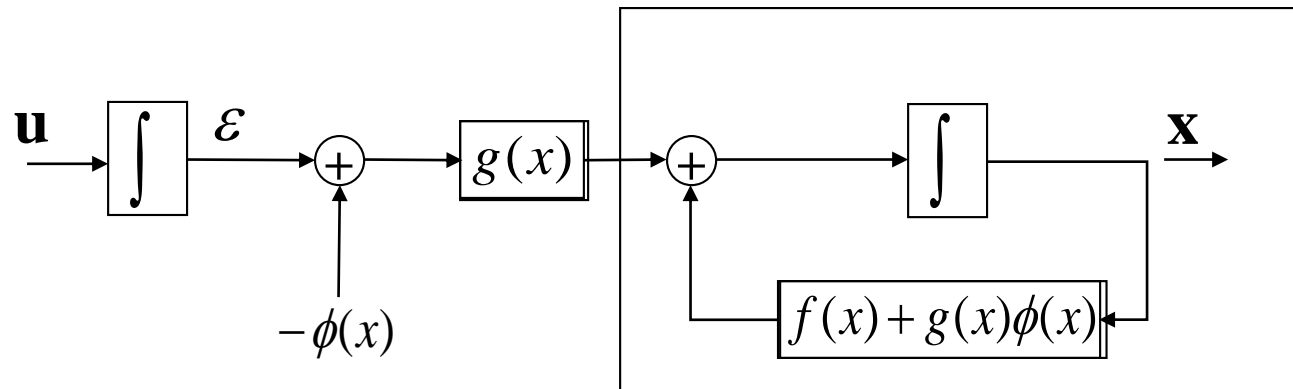
The machinery of State Feedback Linearization could be used to find an alternative output for non-minimum phase acceleration tracking – however the transformations are fairly intractable .

# INTEGRATOR BACKSTEPPING

Given:

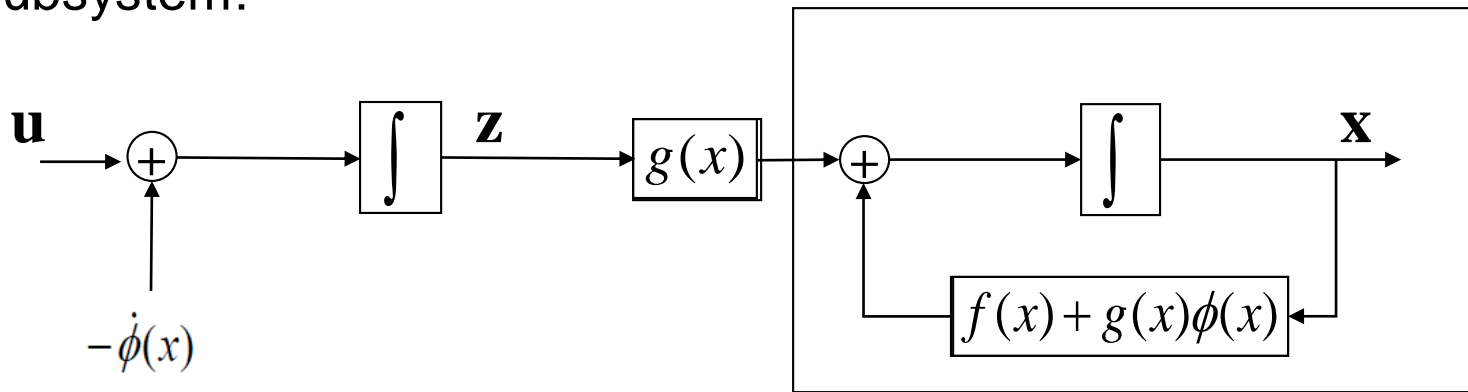


Design a stabilizing “fictitious” control  $\phi(x)$  for the output subsystem:



# INTEGRATOR BACKSTEPPING

Then “backstep” that control through the integrator of the previous subsystem:



It can be seen that a change of variable  $z = \varepsilon - \phi(x)$  is equivalent to adding zero to the original subsystem:

$$\dot{x} = f(x) + g(x)(\varepsilon - \phi(x) + \phi(x))$$

Resulting in an equivalent subsystem:

$$\dot{x} = f(x) + g(x)\phi(x) + g(x)z$$

$$\dot{z} = u - \dot{\phi}(x)$$

# RECURSIVE DESIGN

By **recursive** application of Integrator Backstepping, we can extend the results to higher order systems provided they have the **strict feedback cascaded** form:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n) + g_n(x_1, x_2, \dots, x_n)u\end{aligned}$$

At each step we define a new state variable:

Until the control design equation “pops” out:

$$z_1 = x_1 - x_1^d$$

$$z_2 = \dot{z}_1 - \dot{z}_1^d$$

$$z_3 = \dot{z}_2 - \dot{z}_2^d$$

$\vdots$

$$z_n = \dot{z}_{n-1} - \dot{z}_{n-1}^d$$

$$\underline{0 = \dot{z}_n - \dot{z}_n^d \Rightarrow u = \dots}$$

# RECURSIVE DESIGN

Also at each step, we recursively accumulate the terms of the Lyapunov function:

$$V = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \Lambda + \frac{1}{2} z_n^2$$

For which it can be shown:  $\dot{V}(z) < 0$

## Advantages of Recursive Backstepping Design over Feedback Linearization

1. Transformation is simple (no PDE's to solve)
2. No need to cancel “beneficial” nonlinearities
3. No need to cancel “weak” nonlinearities
4. Can add robust stabilizing terms to overcome uncertainties

# ROBUST RECURSIVE DESIGN

Model the system with uncertainties:

$$\dot{x}_1 = f_1(x_1) + \Delta f_1(x_1) + g_1(x_1)x_2$$

$$\dot{x}_2 = f_2(x_1, x_2) + \Delta f_2(x_1, x_2) + g_2(x_1, x_2)x_3$$

⋮

$$\dot{x}_n = f_n(x_1, x_2, \dots, x_n) + \Delta f_n(x_1, x_2, \dots, x_n) + g_n(x_1, x_2, \dots, x_n)u$$

Which meet the *Generalized Matching Conditions*

Bound the uncertainty that appears in the  $\dot{z}_i$  equations:

$$\|\Delta f(z_i)\| \leq \rho_i$$

Add a robust fictitious control term to the

Such as:

$$z_{i+1} = \dot{z}_i - \dot{z}_i^d + v_i$$
$$v_i = \left(1 - \text{sgn}(z_i)e^{-\sigma_i|z_i|}\right)\rho_i$$

# DYNAMIC RECURSIVE DESIGN

Dynamic Recursive Design extends the Recursive Process to systems which do not meet the strict feedback cascaded form:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, \dots, x_n) + g_1(x_1, x_2, \dots, x_n)u \\ \dot{x}_2 &= f_2(x_1, x_2, \dots, x_n) + g_2(x_1, x_2, \dots, x_n)u \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n) + g_n(x_1, x_2, \dots, x_n)u\end{aligned}$$

By differentiating  $u$   $(n-r)$  times to create additional state variables, such that:

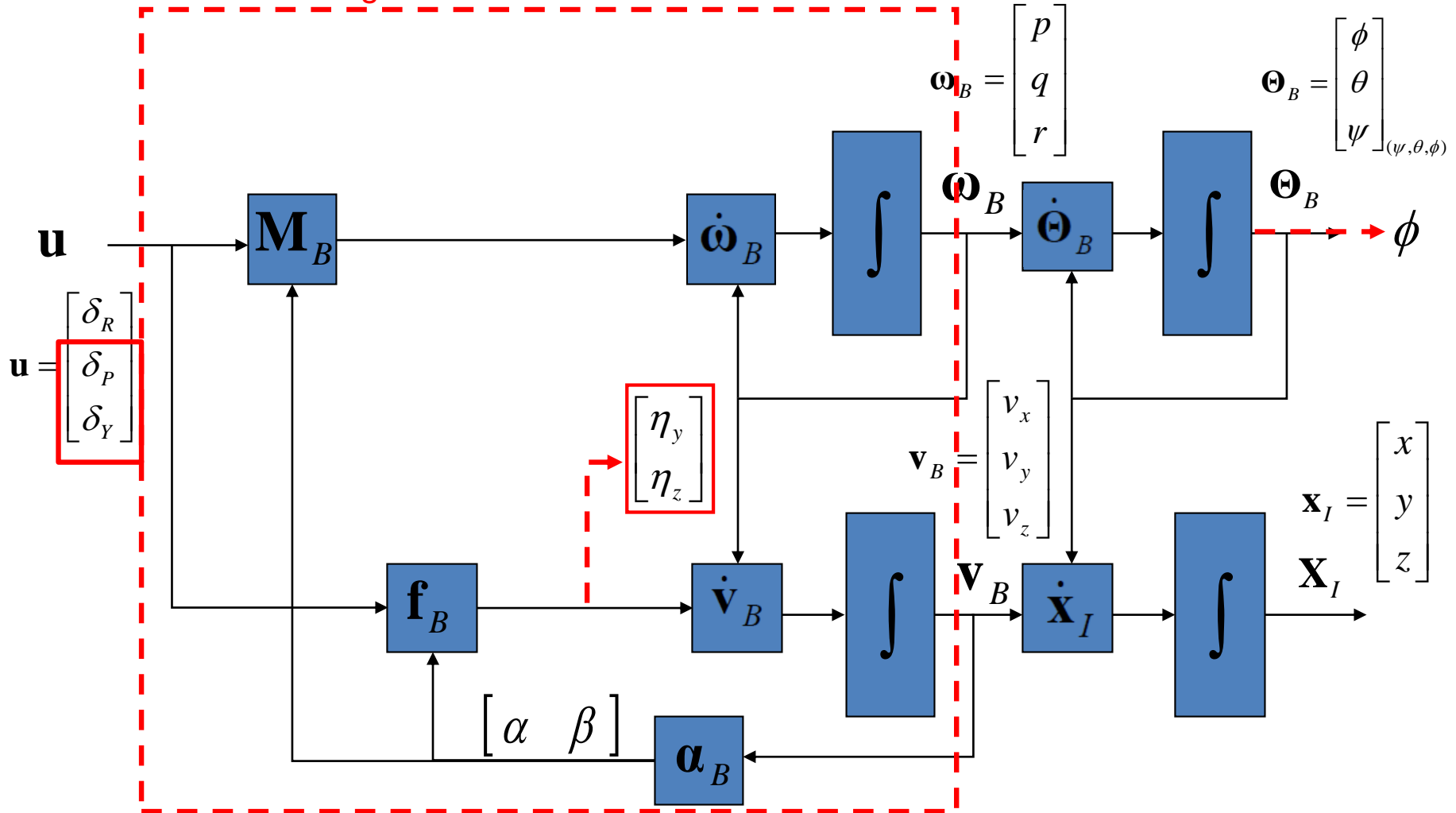
$$z_i = z_i(x, u, \dot{u}, \dots, u^{(n-r-1)})$$

Until the control design that “pops” out is a derivative of  $u$ :

$$\begin{aligned}z_1 &= x_1 - x_1^d \\ z_2 &= \dot{z}_1 - \dot{z}_1^d \\ z_3 &= \dot{z}_2 - \dot{z}_2^d \\ &\vdots \\ z_n &= \dot{z}_{n-1} - \dot{z}_{n-1}^d \\ 0 &= \dot{z}_n - \dot{z}_n^d \Rightarrow u^{(n-r)} = \dots\end{aligned}$$

# 6-DOF RIGID BODY EQUATIONS OF MOTION

## Recursive Design for Pitch/Yaw Acceleration Control





# 6-DOF RIGID BODY EQUATIONS OF MOTION

12 state equations in body coordinates:

$$\dot{\mathbf{v}}_B = \frac{1}{m} (\mathbf{f}_B - \dot{m} \mathbf{v}_B) - \boldsymbol{\omega}_B \otimes \mathbf{v}_B$$

$\mathbf{f}_B =$  External body forces  
 $m =$  mass

$$\dot{\boldsymbol{\omega}}_B = \mathbf{J}_B^{-1} (\mathbf{M}_B - \dot{\mathbf{J}}_B \boldsymbol{\omega}_B - \boldsymbol{\omega}_B \otimes \mathbf{J}_B \boldsymbol{\omega}_B)$$

Using the Euler Rate equations:

$$\dot{\boldsymbol{\Theta}}_B = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}_{(\psi, \theta, \phi)} = \begin{bmatrix} p + \tan \theta (q \sin \phi + r \cos \phi) \\ q \cos \phi - r \sin \phi \\ \frac{1}{\cos \theta} (q \sin \phi + r \cos \phi) \end{bmatrix}$$

$\mathbf{M}_B =$  External body moments

$\mathbf{J}_B =$  Moment of Inertia Matrix

$$\dot{\mathbf{x}}_I = \mathbf{T}_{BI}^{-1} \mathbf{v}_B$$

# 6-DOF RIGID BODY EQUATIONS OF MOTION

The external body forces are computed as functions of:

$$\mathbf{f}_B = \begin{bmatrix} F_x(\alpha, \beta, M, \bar{q}, \delta_p, \delta_y, \delta_r) \\ F_y(\alpha, \beta, M, \bar{q}, \delta_p, \delta_y) \\ F_z(\alpha, \beta, M, \bar{q}, \delta_p, \delta_y) \end{bmatrix} \approx \begin{bmatrix} F_{x_{BT}}(\alpha, \beta, M, \bar{q}) \\ F_{y_{BT}}(\alpha, \beta, M, \bar{q}) \\ F_{z_{BT}}(\alpha, \beta, M, \bar{q}) \end{bmatrix} + \begin{bmatrix} F_{x_i}(\alpha, \beta, M, \bar{q}, \delta_r) \\ F_{y_i}(\alpha, \beta, M, \bar{q}, \delta_y) \\ F_{z_i}(\alpha, \beta, M, \bar{q}, \delta_p) \end{bmatrix}$$

where :

$\alpha$  = Angle of Attack

$\beta$  = Angle of Side Slip

$M$  = Mach Number,

$\bar{q}$  = dynamic pressure

Body Tail + Canard Control Increments

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \tan^{-1}\left(\frac{v_x}{v_z}\right) \\ \sin^{-1}\left(\frac{v_y}{v_T}\right) \end{bmatrix}^T \quad v_T = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$\mathbf{T}_{BI}^{-1} = \mathbf{T}_{BI}^T = \begin{bmatrix} \cos \theta \cos \psi & -\cos \phi \sin \psi + \sin \phi \sin \theta \cos \psi & \sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi \\ \cos \theta \sin \psi & \cos \phi \cos \psi + \sin \phi \sin \theta \sin \psi & -\sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi \\ -\sin \theta & \sin \phi \cos \theta & \cos \phi \cos \theta \end{bmatrix}$$

# Recursive Design for Coupled Pitch/Yaw Controller

Reference: **Dynamic Robust Recursive Control: Theory and Applications**, Ph.D. Dissertation, Richard A. Hull, University of Central Florida, Orlando Florida, 1996.

**First, Define Output Error States:**

$$z_1 = \begin{bmatrix} a_z - a_{zD} \\ a_y - a_{yD} \end{bmatrix}$$

$a_{zD} \equiv$  desired pitch acceleration  
 $a_{yD} \equiv$  desired yaw acceleration  
 $a_z =$  pitch acceleration feedback  
 $a_y =$  yaw acceleration feedback

**Then, recursively define the state transformations:**

$$\begin{aligned} z_1 &= x_1 - x_1^d \\ z_2 &= \dot{z}_1 - \dot{z}_1^d + v_1 \\ 0 &= \dot{z}_2 - \dot{z}_2^d + v_2 \Rightarrow u = \dots \end{aligned}$$

**Where  $v_1, v_2$  represent robust fictitious control terms to overcome bounded uncertainties.**

# Recursive Design for Coupled Pitch/Yaw Controller

Choosing the desired dynamics to map to a stable linear system:

$$\dot{z}_1^d = -k_1 z_1 \quad , \quad \dot{z}_2^d = -k_2 z_2 \quad \text{Design gains: } k_1, k_2$$

The Design Equation (without robust terms) is:

$$\ddot{z}_1 + (k_1 + k_2)\dot{z}_1 + k_1 k_2 z_1 = 0 \quad \text{Where:} \quad \ddot{\mathbf{z}}_1 = \begin{bmatrix} \ddot{a}_z \\ \ddot{a}_y \end{bmatrix}, \quad \dot{\mathbf{z}}_1 = \begin{bmatrix} \dot{a}_z \\ \dot{a}_y \end{bmatrix}$$

Assume that  $\alpha, \beta, \bar{q}, m, v_m$  are known parameters, and define the body normal accelerations to be:

$$\mathbf{a}_B = \begin{bmatrix} a_z \\ a_y \end{bmatrix} = \frac{1}{m} \begin{bmatrix} F_z \\ F_y \end{bmatrix}$$

Where  $F_z$  and  $F_y$  are the aerodynamic forces normal to the body x axis.

We write the Recursive Design Equation for an acceleration controller:

$$\ddot{\mathbf{a}}_B + (k_1 + k_2)\dot{\mathbf{a}}_B + k_1 k_2 (\mathbf{a}_B - \mathbf{a}_B^d) = 0$$

# Recursive Design for Coupled Pitch/Yaw Controller

Now, assume that Mach and Dynamic Pressure are relatively slowly changing parameters and that the normal force contribution from delta dots is negligible, the time differential of the acceleration equations can be represented by:

$$\begin{bmatrix} \dot{a}_z \\ \dot{a}_y \end{bmatrix} \approx \left( \frac{1}{m} \right) \underbrace{\begin{bmatrix} \frac{\partial F_z}{\partial \alpha} & \frac{\partial F_z}{\partial \beta} \\ \frac{\partial F_y}{\partial \alpha} & \frac{\partial F_y}{\partial \beta} \end{bmatrix}}_{\text{Slopes from Aero Functions}} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}$$

Differentiate again to get:

$$\begin{bmatrix} \ddot{a}_z \\ \ddot{a}_y \end{bmatrix} = \left( \frac{1}{m} \right) \underbrace{\begin{bmatrix} \frac{\partial F_z}{\partial \alpha} & \frac{\partial F_z}{\partial \beta} \\ \frac{\partial F_y}{\partial \alpha} & \frac{\partial F_y}{\partial \beta} \end{bmatrix}}_{\text{Slopes from Aero Functions}} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} + \left( \frac{1}{m} \right) \underbrace{\begin{bmatrix} \frac{\partial^2 F_{z_{BT}}}{\partial \alpha^2} & \frac{2\partial^2 F_{z_{BT}}}{\partial \alpha \partial \beta} & \frac{\partial^2 F_{z_{BT}}}{\partial \beta^2} \\ \frac{\partial^2 F_{y_{BT}}}{\partial \alpha^2} & \frac{2\partial^2 F_{y_{BT}}}{\partial \alpha \partial \beta} & \frac{\partial^2 F_{y_{BT}}}{\partial \beta^2} \end{bmatrix}}_{\text{Assume to be zero or negligible}} \begin{bmatrix} \dot{\alpha}^2 \\ \dot{\alpha} \dot{\beta} \\ \dot{\beta}^2 \end{bmatrix} + \dots$$

# Recursive Design for Coupled Pitch/Yaw Controller

Using just the pitch and yaw dynamic equations and assuming  $p = 0$ , we can compute the rate of change in alpha and beta:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 1 & -\sin \alpha \tan \beta \\ 0 & -\cos \alpha \end{bmatrix} \begin{bmatrix} q \\ r \end{bmatrix} + \begin{bmatrix} 0 & \left( \frac{\cos \alpha}{v_m \cos \beta} \right) \\ \left( \frac{\cos \beta}{v_m} \right) & \left( \frac{-\sin \alpha \sin \beta}{v_m} \right) \end{bmatrix} \begin{bmatrix} a_y \\ a_z \end{bmatrix}$$

Then, assuming small angle approximations for alpha and beta:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \underbrace{\begin{bmatrix} q \\ -r \end{bmatrix}}_{\omega_B} + \left( \frac{1}{v_m} \right) \begin{bmatrix} a_z \\ a_y \end{bmatrix}$$

And ...

$$\begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{q} \\ -\dot{r} \end{bmatrix}}_{\dot{\omega}_B} + \left( \frac{1}{v_m} \right) \begin{bmatrix} \dot{a}_z \\ \dot{a}_y \end{bmatrix}$$

# Recursive Design for Coupled Pitch/Yaw Controller

Using approximations for the time derivatives of alpha and beta:

$$\begin{bmatrix} \dot{a}_z \\ \dot{a}_y \end{bmatrix} \approx \left( \frac{1}{m} \right) \begin{bmatrix} \frac{\partial F_z}{\partial \alpha} & \frac{\partial F_z}{\partial \beta} \\ \frac{\partial F_y}{\partial \alpha} & \frac{\partial F_y}{\partial \beta} \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \left( \frac{1}{m} \right) \begin{bmatrix} \frac{\partial F_z}{\partial \alpha} & \frac{\partial F_z}{\partial \beta} \\ \frac{\partial F_y}{\partial \alpha} & \frac{\partial F_y}{\partial \beta} \end{bmatrix} \begin{bmatrix} q + \frac{a_z}{v_m} \\ -r + \frac{a_y}{v_m} \end{bmatrix}$$

And ...

$$\begin{aligned} \begin{bmatrix} \ddot{a}_z \\ \ddot{a}_y \end{bmatrix} &= \left( \frac{1}{m} \right) \begin{bmatrix} \frac{\partial F_z}{\partial \alpha} & \frac{\partial F_z}{\partial \beta} \\ \frac{\partial F_y}{\partial \alpha} & \frac{\partial F_y}{\partial \beta} \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} = \left( \frac{1}{m} \right) \begin{bmatrix} \frac{\partial F_z}{\partial \alpha} & \frac{\partial F_z}{\partial \beta} \\ \frac{\partial F_y}{\partial \alpha} & \frac{\partial F_y}{\partial \beta} \end{bmatrix} \begin{bmatrix} \dot{q} \\ -\dot{r} \end{bmatrix} + \left( \frac{1}{mv_m} \right) \begin{bmatrix} \frac{\partial F_z}{\partial \alpha} & \frac{\partial F_z}{\partial \beta} \\ \frac{\partial F_y}{\partial \alpha} & \frac{\partial F_y}{\partial \beta} \end{bmatrix} \begin{bmatrix} \dot{a}_z \\ \dot{a}_y \end{bmatrix} \\ &= \left( \frac{1}{m} \right) \begin{bmatrix} \frac{\partial F_z}{\partial \alpha} & \frac{\partial F_z}{\partial \beta} \\ \frac{\partial F_y}{\partial \alpha} & \frac{\partial F_y}{\partial \beta} \end{bmatrix} \begin{bmatrix} \dot{q} \\ -\dot{r} \end{bmatrix} + \left( \frac{1}{m^2 v_m} \right) \begin{bmatrix} \frac{\partial F_z}{\partial \alpha} & \frac{\partial F_z}{\partial \beta} \\ \frac{\partial F_y}{\partial \alpha} & \frac{\partial F_y}{\partial \beta} \end{bmatrix}^2 \begin{bmatrix} q + \frac{a_z}{v_m} \\ -r + \frac{a_y}{v_m} \end{bmatrix} \end{aligned}$$

We will solve for the control input to give the required body angular accelerations ...

# Recursive Design for Coupled Pitch/Yaw Controller

Recalling the recursive design equation,

$$\ddot{\mathbf{a}}_B + (k_1 + k_2)\dot{\mathbf{a}}_B + k_1k_2(\mathbf{a}_B - \mathbf{a}^d_B) = 0$$

and using the pitch-yaw acceleration and body rate vectors:

$$\mathbf{a}_B = \begin{bmatrix} a_z \\ a_y \end{bmatrix}, \quad \boldsymbol{\omega}_B = \begin{bmatrix} q \\ -r \end{bmatrix}$$

We can substitute the preceding results into the design equation

$$\left(\frac{1}{m}\right)\mathbf{F}\dot{\boldsymbol{\omega}}_B + \left(\frac{1}{m^2v_m}\right)\mathbf{F}^2\left(\boldsymbol{\omega}_B + \frac{\mathbf{a}_B}{v_m}\right) + (k_1 + k_2)\left(\frac{1}{m}\right)\mathbf{F}\left(\boldsymbol{\omega}_B + \frac{\mathbf{a}_B}{v_m}\right) + k_1k_2(\mathbf{a}_B - \mathbf{a}^d_B) = 0$$

Where we have defined

$$\mathbf{F} = \begin{bmatrix} \frac{\partial F_z}{\partial \alpha} & \frac{\partial F_z}{\partial \beta} \\ \frac{\partial F_y}{\partial \alpha} & \frac{\partial F_y}{\partial \beta} \end{bmatrix}$$



# Recursive Design for Coupled Pitch/Yaw Controller

Now, provided  $\mathbf{F}$  is invertible, we can solve the recursive design equation, for the required body angular accelerations:

$$\dot{\boldsymbol{\omega}}_B^R = -\left(\frac{1}{m\nu_m}\right)\mathbf{F}\left(\boldsymbol{\omega}_B + \frac{\mathbf{a}_B}{\nu_m}\right) - (k_1 + k_2)\left(\boldsymbol{\omega}_B + \frac{\mathbf{a}_B}{\nu_m}\right) - k_1k_2m\mathbf{F}^{-1}(\mathbf{a}_B - \mathbf{a}_B^d)$$

Equating to the body moment equations:

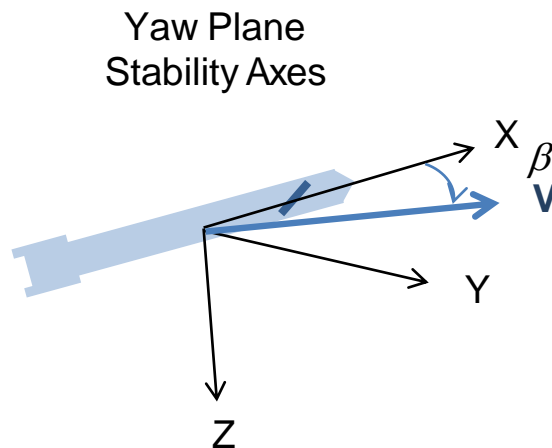
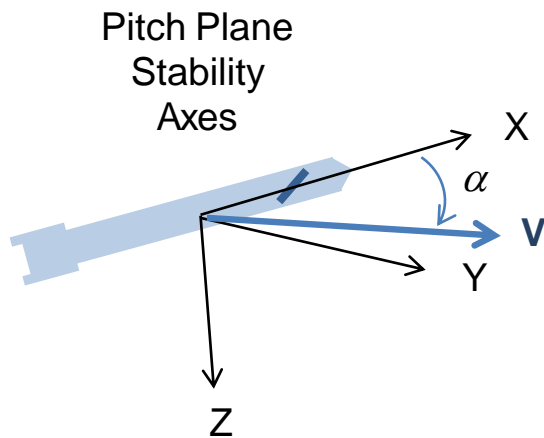
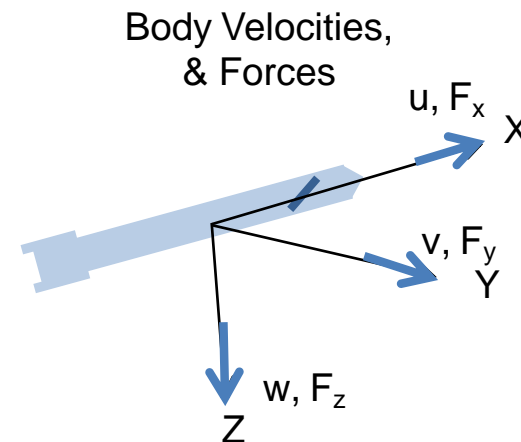
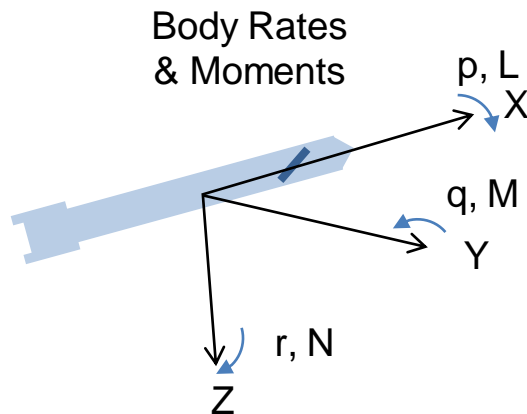
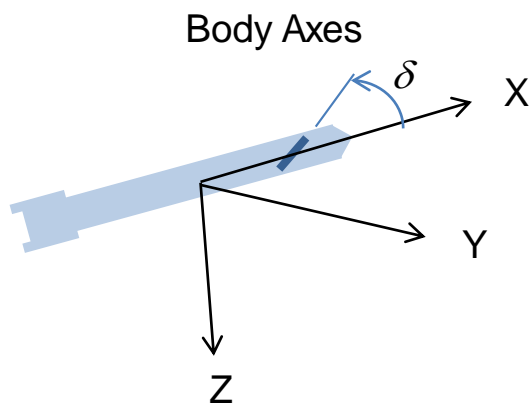
$$\dot{\boldsymbol{\omega}}_B = \mathbf{J}_B^{-1}(\mathbf{M}_B - \boldsymbol{\omega}_B \otimes \mathbf{J}_B \boldsymbol{\omega}_B)$$

We can solve for the body moments required to give the desired body acceleration response:

$$\mathbf{M}_B^R = \mathbf{J}_B \dot{\boldsymbol{\omega}}_B^R + \boldsymbol{\omega}_B \otimes \mathbf{J}_B \boldsymbol{\omega}_B$$

Then, we “invert” the aerodynamic moment equations to find the pitch and yaw control input deflections that will give the required body moments above.

# Conventions for Aerodynamic Definitions



where :

$\alpha$  = Angle of Attack

$\beta$  = Angle of Side Slip

$\delta$  = Canard Deflection,

$p, q, r$  = Body Rates

$u, v, w$  = Body Velocities

$L, M, N$  = Body Moments

$F_x, F_y, F_z$  = Body Forces

# Hypothetical Projectile Aerodynamics Model

Define aerodynamic forces and moments in the pitch and yaw axes in terms of alpha, beta, and the pitch and yaw control deflections as:

$$F_z = \bar{q} S_{ref} C_Z(\alpha, \beta, \delta_p) \quad \text{Aero Force along body z axis}$$

$$F_y = \bar{q} S_{ref} C_Y(\alpha, \beta, \delta_Y) \quad \text{Aero Force along body y axis}$$

$$m = \bar{q} S_{ref} l_{ref} C_m(\alpha, \beta, \delta_p) \quad \text{Aero Pitching Moment}$$

$$n = \bar{q} S_{ref} l_{ref} C_n(\alpha, \beta, \delta_Y) \quad \text{Aero Yawing Moment}$$

Where:  $\bar{q}$  = dynamic pressure,  $S_{ref}$  = reference area,  $l_{ref}$  = reference length  
Assume aerodynamic coefficients are the following functions of alpha, beta, pitch and yaw control deflections (in degrees):

$$C_Z(\alpha, \beta, \delta_p) = .000103\alpha^3 - .00945\alpha|\alpha| - .017\alpha - .00055\alpha^2\beta - .034\delta_p$$

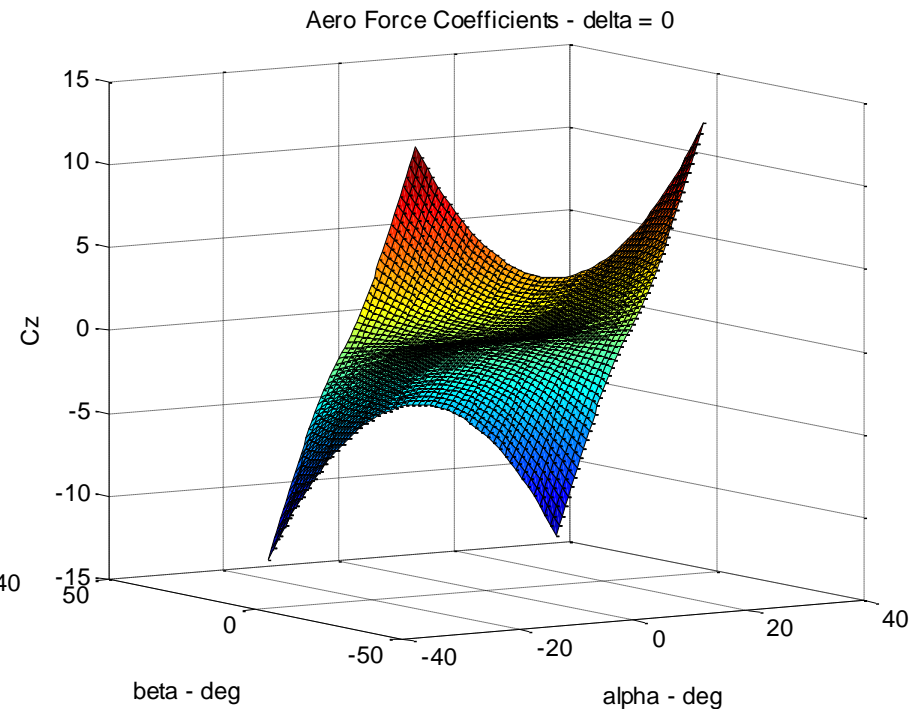
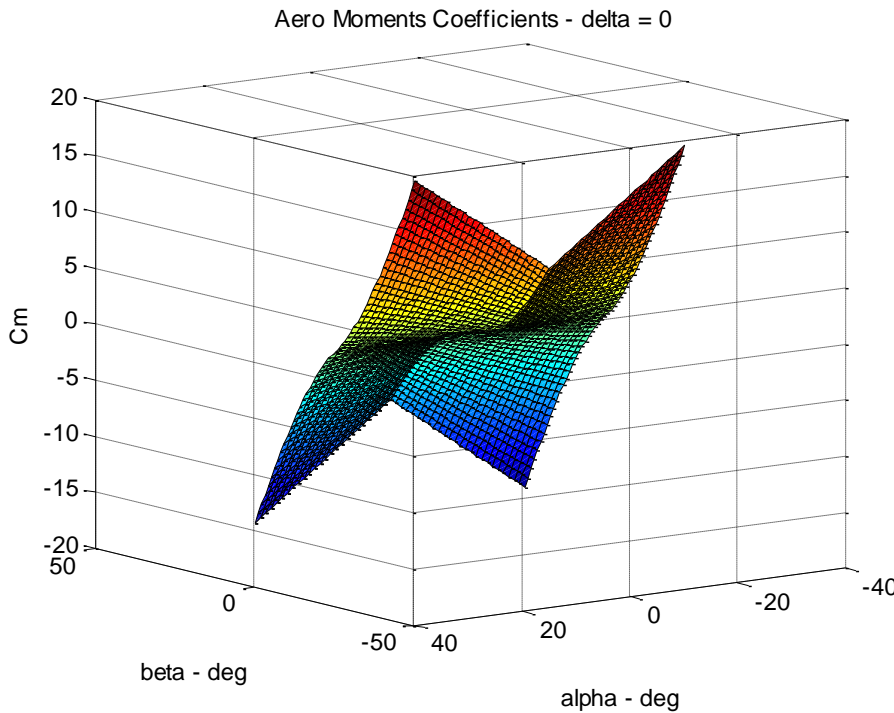
$$C_Y(\alpha, \beta, \delta_Y) = .000103\beta^3 - .00945\beta|\beta| - .017\beta - .00055\beta^2\alpha - .034\delta_Y$$

$$C_m(\alpha, \beta, \delta_p) = .000215\alpha^3 - .0195\alpha|\alpha| + .051\alpha - .015\alpha|\beta| + .700\delta_p$$

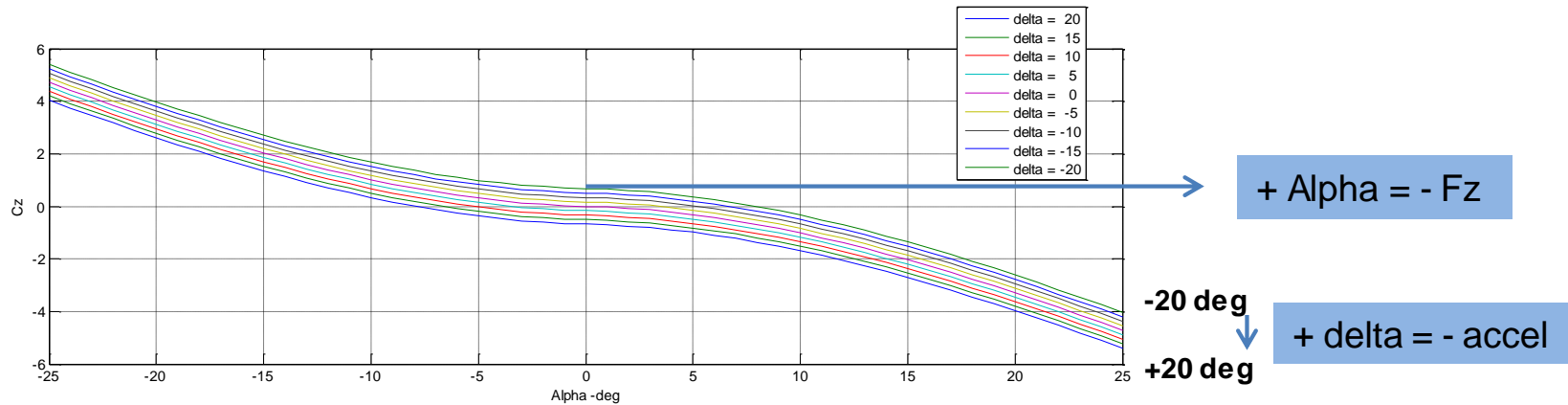
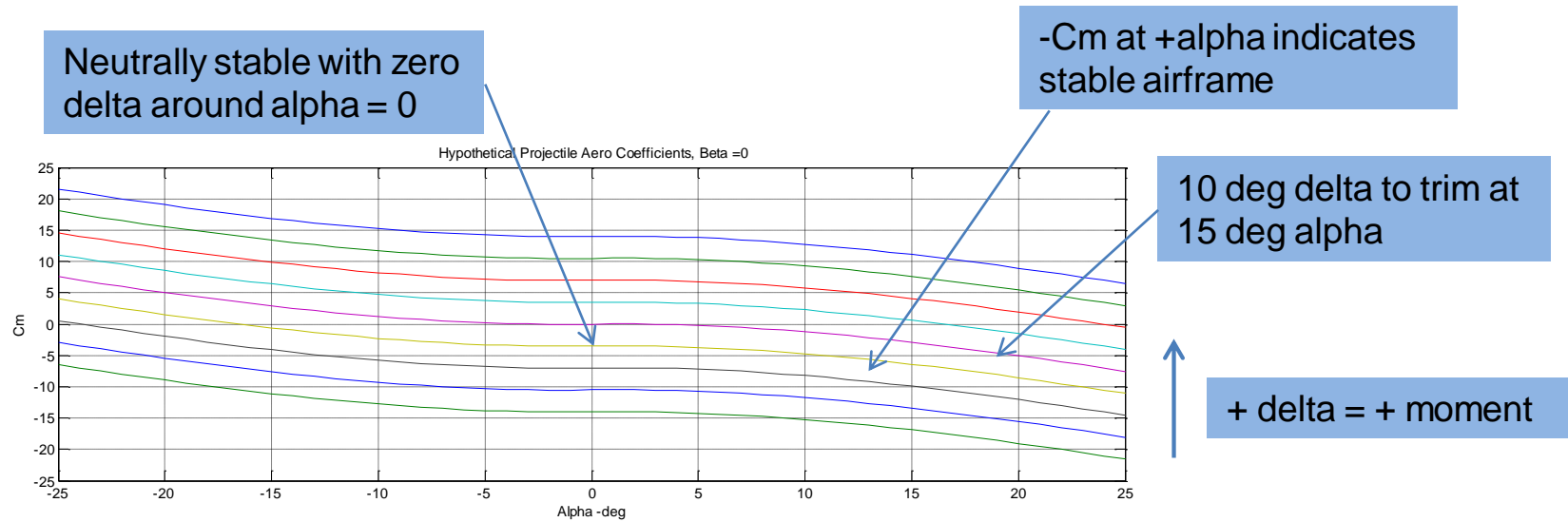
$$C_n(\alpha, \beta, \delta_Y) = .000215\beta^3 - .0195\beta|\beta| + .051\beta - .015\beta|\alpha| + .700\delta_Y$$

Coupled  
Nonlinear  
Equations  
in Alpha  
and Beta

# Aerodynamic Force and Moment Coefficients for a Hypothetical Projectile as Function of Alpha, Beta for Delta = 0



# Aerodynamic Force and Moment Coefficients for a Hypothetical Projectile as Function of Alpha, Beta and Delta



# Recursive Pitch/Yaw Controller for the Hypothetical Projectile

Using the Force Coefficient Polynomials:

$$C_Z(\alpha, \beta, \delta_P) = a_1\alpha^3 + a_2\alpha|\alpha| + a_3\alpha + a_4\alpha^2\beta + b_1\delta_P$$

$$C_Y(\alpha, \beta, \delta_Y) = a_1\beta^3 + a_2\beta|\beta| + a_3\beta + a_4\beta^2\alpha + b_1\delta_Y$$

where:  $a_1 = 0.000103$ ,  $a_2 = -0.00945$ ,  $a_3 = -0.017$ ,  $a_4 = -0.00055$ ,  $b_1 = -0.034$

We can find the required “aero slopes” matrix:

$$\mathbf{F} = \begin{bmatrix} \frac{\partial F_z}{\partial \alpha} & \frac{\partial F_z}{\partial \beta} \\ \frac{\partial F_y}{\partial \alpha} & \frac{\partial F_y}{\partial \beta} \end{bmatrix} = \bar{q} S_{ref} \begin{bmatrix} \frac{\partial C_z}{\partial \alpha} & \frac{\partial C_z}{\partial \beta} \\ \frac{\partial C_y}{\partial \alpha} & \frac{\partial C_y}{\partial \beta} \end{bmatrix} \left( \frac{180}{\pi} \right) \leftarrow \text{because } (\alpha, \beta) \text{ are in degrees!}$$

From the coefficient polynomials:

$$\frac{\partial C_z}{\partial \alpha} = 3a_1\alpha^2 + 2a_2\alpha \text{sign}(\alpha) + a_3 + 2a_4\alpha\beta \quad , \quad \frac{\partial C_z}{\partial \beta} = a_4\alpha^2$$

$$\frac{\partial C_y}{\partial \alpha} = a_4\beta^2 \quad , \quad \frac{\partial C_y}{\partial \beta} = 3a_1\beta^2 + 2a_2\beta \text{sign}(\beta) + a_3 + 2a_4\beta\alpha$$

# Recursive Pitch/Yaw Controller for the Hypothetical Projectile

Using  $F$ , we compute the required body angular rates and moments:

$$\dot{\omega}^R_B = -\left(\frac{1}{mv_m}\right)\mathbf{F}\left(\omega_B + \frac{\mathbf{a}_B}{v_m}\right) - (k_1 + k_2)\left(\omega_B + \frac{\mathbf{a}_B}{v_m}\right) - k_1k_2m\mathbf{F}^{-1}(\mathbf{a}_B - \mathbf{a}^d_B)$$

$$\mathbf{M}^R_B = \mathbf{J}_B \dot{\omega}^R_B + \omega_B \otimes \mathbf{J}_B \omega_B$$

Ignore – pitch and yaw components are zero if  $p = 0$

where:  $\mathbf{a}_B = \begin{bmatrix} a_Z \\ a_Y \end{bmatrix}$ ,  $\omega_B = \begin{bmatrix} q \\ -r \end{bmatrix}$  are from the accelerometer and gyro feedbacks

the following parameters are estimated in real-time:

$\alpha, \beta$  = Angle of Attack and Angle of Side Slip

$\bar{q}$  = dynamic pressure

$v_m$  = velocity of projectile

$m$  = mass of projectile

Note – in the autopilot we have let  $w_b(2) = -r$ , therefore reverse the sign of the required yaw moment!

$$\mathbf{J}_B = \text{pitch/yaw moments of intetial} = \begin{bmatrix} J_{zz} & J_{yz} \\ J_{yz} & J_{yy} \end{bmatrix}$$

Inertia Coupling Included

And  $k_1, k_2$  are gains chosen by the designer to place the poles of the feedback linearized system.

# Recursive Design Applied to the Hypothetical Projectile

Finally, we use the aerodynamic moment equations:

$$C_m(\alpha, \beta, \delta_p) = c_1\alpha^3 + c_2\alpha|\alpha| + c_3\alpha + c_4\alpha|\beta| + d_1\delta_p$$

$$C_n(\alpha, \beta, \delta_Y) = c_1\beta^3 + c_2\beta|\beta| + c_3\beta + c_4\beta|\alpha| + d_1\delta_Y$$

Note: Pitch and Yaw Control Contributions May Not be De-coupled in the Real World!

where :  $c_1 = 0.000215$ ,  $c_2 = -0.0195$ ,  $c_3 = 0.051$ ,  $c_4 = -0.015$ ,  $d_1 = 0.700$

to solve for the pitch and yaw control deflections needed to give the required pitch and yaw moments:

$$\mathbf{M}_B^R = \begin{bmatrix} M_p^R \\ M_Y^R \end{bmatrix} = \bar{q}S_{ref}l_{ref} \begin{bmatrix} C_m(\alpha, \beta, \delta_p) \\ C_n(\alpha, \beta, \delta_Y) \end{bmatrix}$$

so the pitch and yaw control deflection commands are :

$$\delta_p = \frac{1}{d_1} \left[ \frac{1}{\bar{q}S_{ref}l_{ref}} M_p^R - (c_1\alpha^3 + c_2\alpha|\alpha| + c_3\alpha + c_4\alpha|\beta|) \right]$$

$$\delta_Y = \frac{1}{d_1} \left[ -\frac{1}{\bar{q}S_{ref}l_{ref}} M_Y^R - (c_1\beta^3 + c_2\beta|\beta| + c_3\beta + c_4\beta|\alpha|) \right]$$

Note – in the autopilot we have let  $w\_b(2) = -r$  , therefore reverse the sign of the required yaw moment!

where again the parameters :  $\alpha$ ,  $\beta$ ,  $\bar{q}$ , are estimated in real - time, and  $S_{ref}$  and  $l_{ref}$  are the known reference area and length.



# Hypothetical Projectile Parameters

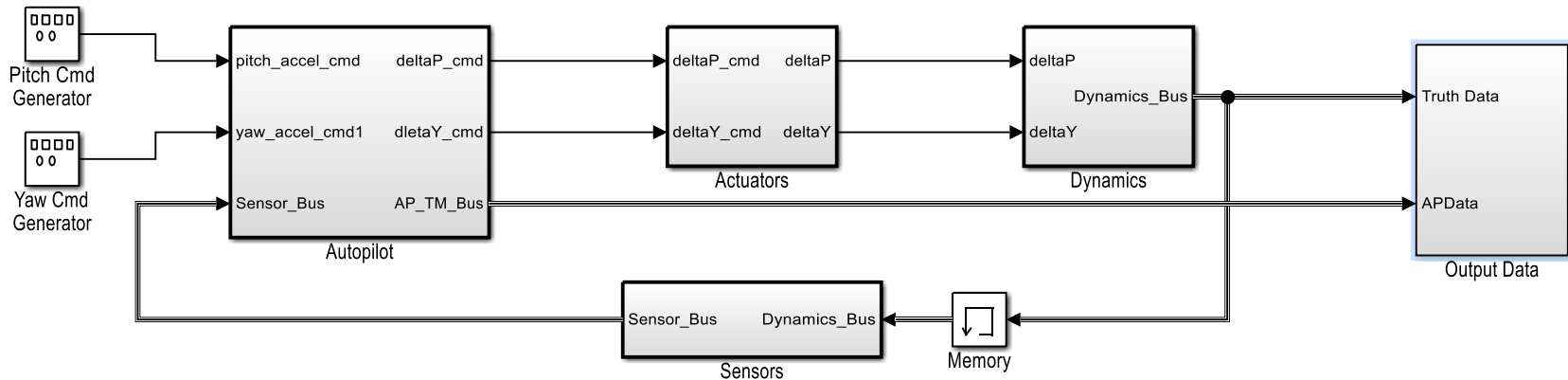
A totally fictitious configuration intended as a model to study guidance and control issues for guided projectiles.

## Model Parameters

Parameter	Value	Units
Diameter	140	mm
Mass	50	kg
Xcg (from nose)	1.0	m
Ixx	0.200	Kg-m <sup>2</sup>
Iyy = Izz	18.00	Kg-m <sup>2</sup>
Iyz = Izy	0.50	Kg-m <sup>2</sup>
Sref (aero reference area)	0.0154	m <sup>2</sup>
Lref (aero reference length)	1.0	m
MRC (aero model ref center)	1.0	m

# MATLAB/Simulink Simulation

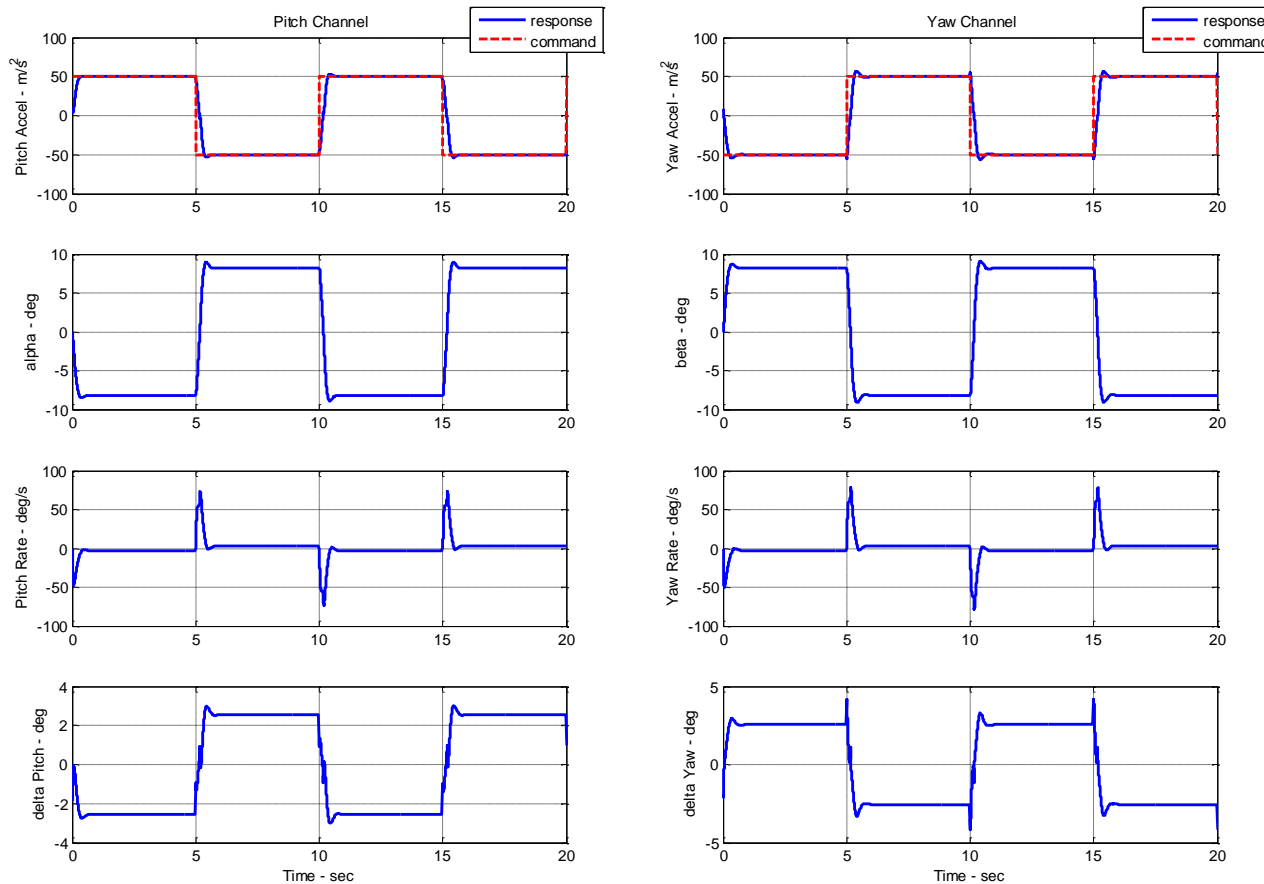
Hypothetical Projectile  
Simplified Pitch/Yaw Controller Simulation  
- Rich Hull -



Unclassified  
This Document Does Not Contain Any Export Controlled Technical Data

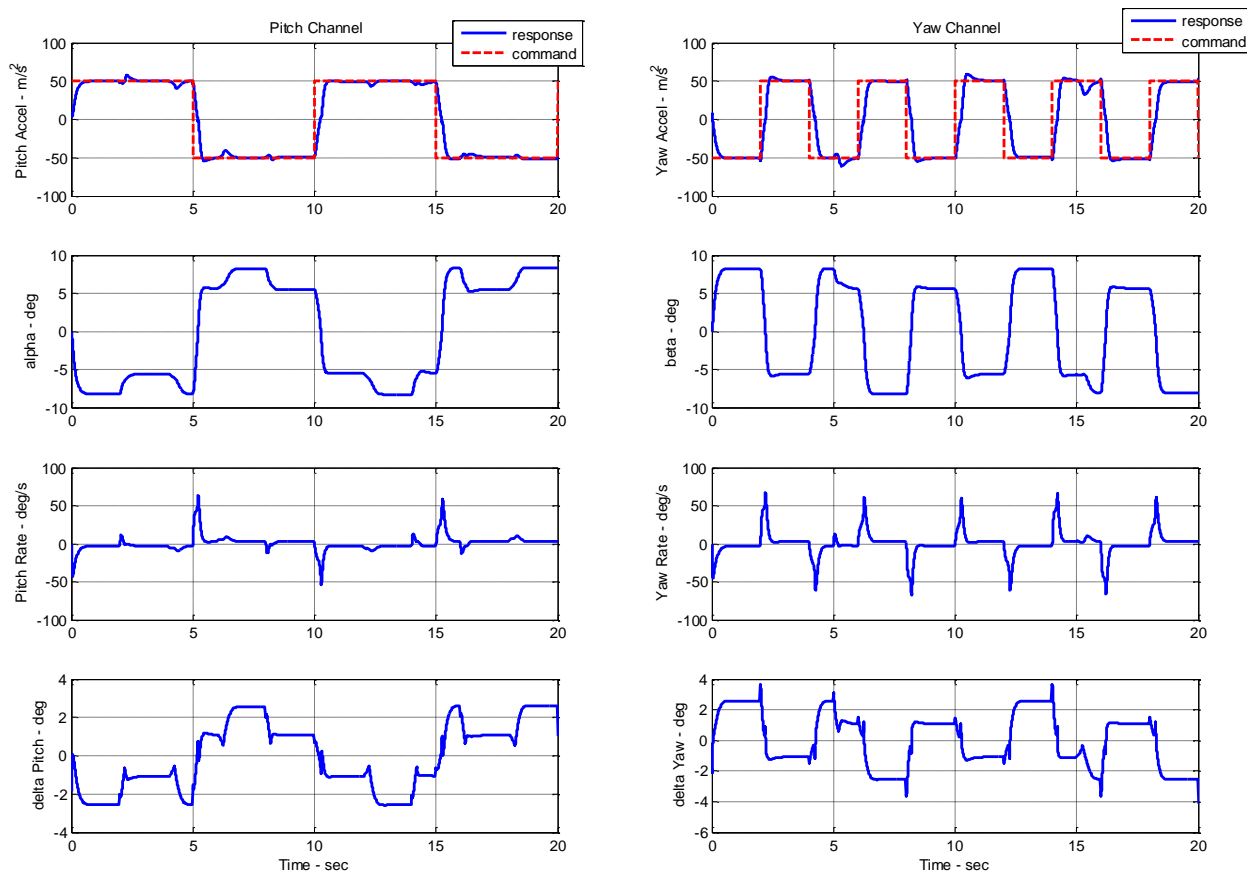
# MATLAB/Simulink Simulation Results

## Simultaneous Pitch and Yaw Step Commands Full Coupled Nonlinear Aero Model with Inertia Coupling



# MATLAB/Simulink Simulation Results

## Simultaneous Pitch and Yaw Step Commands at Different Frequencies Full Coupled Nonlinear Aero Model with Inertia Coupling



# Summary and Conclusions

- Precision Guided Projectiles Pose Challenging Guidance, Navigation and Control Problems
- GENEX Guidance Law Provides an Analytical Solution to the Optimal Guidance Problem
- Nonlinear Recursive Control Design Approach Demonstrated to Get Analytical Solution for Pitch/Yaw Autopilot for a Hypothetical Projectile with Coupled Nonlinear Aerodynamics and Off-Diagonal Inertia Coupling Terms.
  - Method requires computation of “slopes” of aerodynamic force functions, and inverse of aero moment functions
  - Good tracking control can be achieved without addition of integrators to controller
  - Good method for getting quick “simulation quality” controller
  - Problems may occur if aerodynamic partial derivatives matrix becomes ill-conditioned
  - Additional terms can be added to provide robustness to bounded uncertainties and un-modelled nonlinear dynamics

# REFERENCES

1. Ernest J. Ohlmeyer and Craig A. Phillips, "Generalized Vector Explicit Guidance", *AIAA Journal of Guidance, Control and Dynamics*, Vol. 29, No. 2, March-April 2006
2. Buffington, J. M., Enns, D. F., and Teel, A. R., "Control Allocation and Zero Dynamics, AIAA Guidance, Navigation, and Control Conference, San Diego, CA, Aug. 1996.
3. Colgren, R., and Enns, D, "Dynamic Inversion Applied to the F-117A", AIAA Modeling and Simulation Technologies Conference, New Orleans, LA, Aug. 1997.
4. Hull, R. A., "Dynamic Robust Recursive Control Design and Its Application to a Nonlinear Missile Autopilot," Proceedings of the 1997 American Control Conference, Vol. I, pp. 833-837, 1997.
5. Buffington, J. M. , and Sparks, A. G., "Comparison of Dynamic Inversion and LPV Tailless Flight Control Law Designs", Proceedings of the American Control Conference, June, 1998.
6. Hull, R. A., Ham, C., and Johnson, R. W., Systematic Design of Attitude Control System for a Satellite in a Circular Orbit with Guaranteed Performance and Stability, Proceedings of the 14<sup>th</sup> Annual AIAA/Utah State University Small Satellite Conference, August, 2000.
7. McFarland, M. B. and Hogue, S. M., " Robustness of a Nonlinear Missile Autopilot Designed Using Dynamic Inversion", AIAA 2000-3970.
8. Slotine, J.-J. E., and Li, W., *Applied Nonlinear Control*, Prentice Hall, 1991.
9. Krstic, M., Kanellakopoulos, I., Kokotovic, P., *Nonlinear and Adaptive Control Design*, John Wiley & Sons, 1995.
10. Qu, Z., *Robust Control of Nonlinear Uncertain Systems*, John Wiley & Sons, 1998.
11. Khalil, H. K., *Nonlinear Systems*, Third Edition, Prentice Hall, 2002.