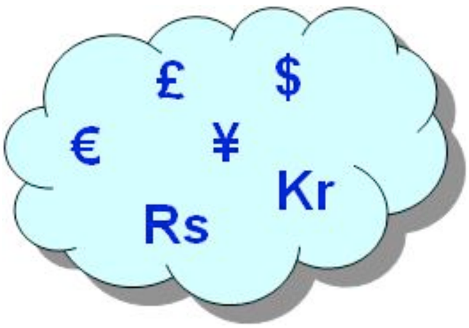
Estimating Heavy-Tailed Distributions in Finance



Financial engineering is about risk assessment and risk management. At the individual and institutional levels, the key objective is to maximize the expected return on one's investment portfolio while staying within the limits of acceptable risk.

Completely eliminating risk is impossible; for example, a bank settles for bounding risk at a 99% (or some other preselected) level, ensuring that the reserves on hand are sufficient to meet all contingencies with a probability of 0.99. These probability computations are based on historical observations—and therein lies the problem.

Crises are caused not by run-of-the-mill events, but by "extreme events"—sometimes by a confluence of extreme events referred to as a "perfect storm." So an important part of financial engineering is estimating "tail probabilities," that is, probabilities of very rare events. It is safe to say that most existing methods for fitting probability distributions to observed data are poorly suited for estimating tail probabilities. This is the challenge that provides an opportunity for control scientists and engineers.

Value at Risk (VaR)

The financial industry uses the concept of VaR (value at risk) as a metric to quantify the risk of a "position" or investment portfolio. The 1% VaR is the 99th percentile of the probability distribution function (of an individual or institutional portfolio), or equivalently, the 1st percentile of the complementary distribution function.

For example, if a portfolio of stocks has a one-day 1% VaR of $1 million, the probability is 0.01 that the portfolio will fall in value by more than $1 million over a one-day period if there is no trading. Obviously, estimating VaR accurately is crucial for financial institutions.

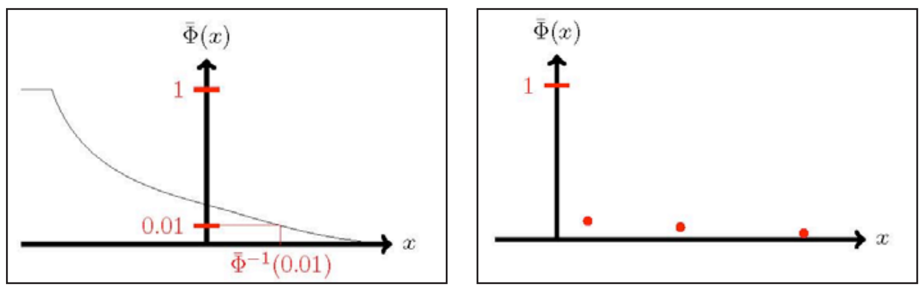


Illustration of the value at risk concept. *The plot on the left shows the complementary probability distribution of the potential loss and the VaR at the 0.01 (1%) level—the loss will be ≥ $\bar{\Phi}^{-1}$ (0.01) with probability of 0.01. The plot on the right shows the three most extreme points out of 250 (one year's data), illustrating the difficulty of estimating the 1% VaR (diagram not to scale).*

Estimating VaR

The complementary distribution function $\bar{\Phi}(\cdot)$ is often estimated using historical records. This approach poses two difficulties:

- An inadequate number of observations, and
- Improper modeling assumptions.

When historical records are used to estimate $\bar{\Phi}(\cdot)$, often a standard distribution function (such as Gaussian, Laplacian, or Pareto) is fitted to observations. Although these may give a good approximation, the key is to obtain a good fit for the "tail" of the distribution because that is where it is most crucial to estimate the risk correctly.

With daily closing records stretching over one year, only 250 data points exist for each random variable; so only 12.5 data points are available to estimate $\bar{\Phi}^{-1}$ (0.05), and a mere 2.5 data points are available to estimate $\bar{\Phi}^{-1}$ (0.01). Going back over periods longer than one year is risky as the process statistics are nonstationary and will have changed. Enlarging the number of samples by pooling data from multiple sources, such as prices of multiple stocks or multiple commodities, is also dangerous if these measurements are highly correlated; the apparent multiplicity of samples is then illusory.

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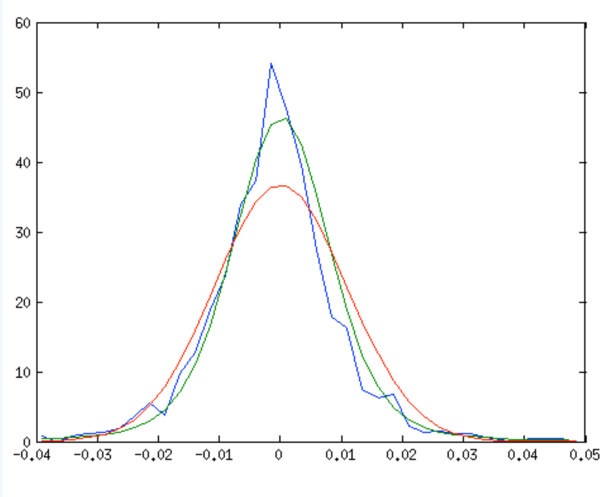
Heavy-Tailed Random Variables

Much of financial engineering is based on so-called "complete markets" and on the use of the Black-Scholes formula. A complete market is one in which every price movement can be "replicated" by a hedging strategy—an unrealistic assumption. A closely related assumption is that asset prices follow a log-normal distribution, or in other words, the daily fluctuations in prices viewed as percentage changes follow a Gaussian distribution. On the contrary, studies of actual asset prices show that they do not follow a log-normal distribution.

Long-term averages of asset returns can be shown to follow a "stable" distribution. Each stable distribution has an associated "exponent" $0 < \alpha \le 2$. The Gaussian is the only stable distribution with finite variance and has $\alpha = 2$. All other stable distributions have $\alpha < 2$ and have infinite variance. If $\alpha < 1$, then even the mean can be infinite, but this situation rarely arises with actual financial data. Such random variables are said to be "heavy-tailed." Moreover, as shown in the figure to the right, real asset movements are better approximated by stable distributions with α well below the critical value of 2. Note that the smaller the value of α, the more slowly the tails decay and the greater the scope for error when Gaussian approximations are used.

Averages of heavy-tailed random variables still follow the law of large numbers (the average converges in probability to the true mean as the number of samples increases) but do not follow the central limit theorem (fluctuations about the true mean are not necessarily Gaussian). In fact, large excursions about the mean are far more "bursty" with heavy-tailed random variables. In short, "rare" events are not as rare as log-normal models would predict. This may be one reason why large swings (ten or more standard deviations when log-normal approximations are used) are far more frequent than a log-normal model would predict.



Daily Returns of the Dow Jones Industrial Average: *The plot shows the daily DJIA fractional returns from January 2000 to March 2007—1,833 samples in all. The green curve, with α = 1.6819, is the best stable fit and fits the data far better than a Gaussian (red curve, α = 2). Note that the best stable fit is also skewed, with negative returns more prevalent than positive returns. Skewed stable distributions are represented by a nonzero value for a second parameter, β (equal to −0.0651 here).*

Other Applications for Estimation of Heavy-Tailed Distributions

In addition to financial engineering, heavy-tailed distributions arise in numerous other applications. Examples include extremes in weather (e.g., rainfall) and Internet traffic. Because of their asymptotic behavior, heavy-tailed stable distributions are also referred to as "power laws."

Role for Control Engineers

Several engineers trained in stochastic control and filtering have made the transition to financial engineering, both in academia and industry. The problems of estimating parameters and signal models are common to both areas. Moreover, dealing with heavy-tailed random variables, the need for which is only now being appreciated, requires sound training in probability theory of the kind imparted to control engineers.

For more information: S.T. Rachev (Ed.), Handbook of Heavy-Tailed Distributions in Finance, Elsevier/North Holland, 2003; W.E. Leland et al., On the self-similar nature of ethernet traffic, IEEE/ACM Trans. Networking, 2(1), 1-15, 1994.