Development and Applications of Multirate Digital Control

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ABSTRACT: Multiple sample rate digital control systems are of prominent interest in current control research, development, and applications. Modern aerospace vehicles and systems are described by high-order dynamic models which typically include phenomena covering a wide range of characteristic frequencies and instrumentation measurements available at multiple rates. A multirate control structure allows the designer to accommodate multiple information rates and implement required control computations within the finite computational capabilities of an onboard computer.

In this paper the historical development, representative design approaches, and example applications of multirate digital control are outlined. A brief survey of traditional design approaches and techniques currently in development is presented. Potential future areas of research and application of multirate control are suggested and discussed.

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Introduction
Multirate digital control is a significant area of current research and application that is motivated by practical implementation needs. The motivation for multirate control has traditionally been in aerospace applications where guidance and control laws must be designed to accommodate multiple rates of sensor measurements and finite throughput capabilities of onboard computers. Multirate design techniques should soon find further utility in control applications for highly distributed systems, such as communication networks, and power-plant/power-distribution networks where the characteristic frequencies and time-constants of a local station’s dynamics may differ significantly from those of the network as a whole.

The historical development of multirate control is outlined in the section “Historical Background.” A survey of four general approaches to multirate design is presented in the section “Design Techniques.” In the section “Example Multirate Control Systems,” examples of currently operational multirate systems are discussed. The paper is summarized and potential future areas of research and application of multirate control are suggested in the final section.

Historical Background
The sheer volume of literature that exists on multirate control techniques underscores the importance of the area and the challenge it presents as a research topic. For example, a literature survey included in a recent paper by Walton [1] notes the contributions of over 50 technical papers related to multirate control techniques. Further, it would be reasonable to suspect that practitioners of digital control have produced an immense body of unpublished “methods that work” in developing and implementing practical systems. In this paper, the origins and evolution of multirate control techniques are summarized briefly, noting those contributions that have provided the basis for the more popular techniques now in use.

A historical overview of digital control development is presented in Fig. 1.* The field of digital control, or more precisely, sampled

*Figure 1 was adapted from a similar chart presented in a thesis by Amit [2].

Fig. 1. Development of multirate digital control.

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data control, originated in radar applications during World War II. Because the rotating antenna of a radar system illuminates a target only intermittently, early radar-aided tracking and fire-control systems had to be designed to utilize data in sampled form. Methods for effective design of control systems using sampled data were under initial development during the late 1940's, and multirate systems theory followed these efforts in the early 1950's.

Initially, researchers developed multirate techniques as a method of evaluating more conventional types of controllers such as continuous systems and single-rate sampled data systems. For example, one could study the inter-sample behavior of a signal or output of a single rate control system by introducing a "phantom sampler" (i.e., a fictitious sampler that operates at a rate some integer ratio higher than that of the controller). A significant early contribution to this general method of analysis, known as frequency decomposition, was made by Sklansky and Ragazzini [3] who described the use of this technique in error-sampled control system development. In the late 1950's, Ragazzini and Franklin [4] published a textbook that described both this technique and the closely related switch decomposition technique. Friedland [5] later related the frequency decomposition technique to periodically-varying control structures, followed by contributions of Coffey and Williams and Boykin and Frazier which dealt with the analysis of multiloop, multirate control structures (multiloop referring to a feedback control structure having nested single-input/single-output compensating elements).

Shortly following the origin of the frequency decomposition technique, a similar frequency domain technique known as switch decomposition was developed. Researchers had begun to see the potential value of multirate systems beyond being a technique for analyzing single-rate systems; switch decomposition seemed a "natural" approach to developing such systems. The switch-decomposition technique attributed to Kranc [6], provided a means of representing a multirate control structure as an equivalent single-rate controller; this representation accomplished, the controller could be designed and analyzed using existing single-rate techniques. In the late 1960's, Jury [7] showed an equivalence of the switch decomposition technique and the frequency decomposition technique. Recently, Whitbeck has developed a vector form of the switch decomposition technique and applied it to various problems in flight control [8–10].

Time-domain methods of multirate stability analysis and design were initiated by Kalman and Bertram [11] with the publication of their state space stability analysis technique in 1959. This paper made a major contribution in showing the power and flexibility of state space techniques in characterizing many types of sampled data control systems, including time-varying systems. Apparently, little significant work was initiated to build on this work for nearly fifteen years. Barry [12] published a paper in 1975 in which he described the design of a multirate regulator and showed that its performance was superior to a single-rate regulator having the same base (slow) sample rate. During 1979–81, researchers at The Analytic Sciences Corporation (TASC) developed a new multirate control design technique based on an optimal estimation and control formulation. This research [13–16] included mathematical formulation of the design problem, development of computational design techniques, and applications of these techniques to flight control examples. Essentially in parallel with the work at TASC, Amit and Powell [2] independently investigated a similar optimal control formulation; their work resulted in, among other things, some practical considerations for implementing multirate control laws and a highly efficient method for solving periodic Ricatti equations related to the design techniques.

Design Techniques

Currently popular techniques and promising new approaches are summarized briefly in this section. The "right" technique for multirate design is a matter of discretion on the part of the designer, depending on the objectives of the application and the training of the design engineer.

Transform Approximation of a Continuous-Time Design

The transform approach to multirate design is probably the technique most widely known to design practitioners, owing to its simplicity and early discovery. In using this technique, the designer first establishes a base design in continuous-time that meets desired specifications, then converts this continuous-time design to a difference-equation form through the Tustin transformation or one of its variants (e.g., ref. 17, page 342). Sample rates for the various system compensators are chosen according to bandwidth; i.e., lower sample rates are used in the mechanization of low-bandwidth compensators.

The Tustin transform in its basic form is given mathematically by

\[ s = \frac{2z - 1}{Tz + 1} \]  

(1)

The designer uses the expression given in eq. 1 to convert a desired continuous-time control law into a discrete-time control law by direct substitution. In effect, the Tustin transform approximates the integration functions of the continuous-time controller by a simple numerical integration technique.

A typical application of the Tustin transform technique is illustrated for the system shown in Fig. 2. Here \( G_c(s) \) represents the dynamics of an electro-mechanical alignment mirror having a parasitic structural mode (i.e., a lightly damped, high frequency mode). The continuous feedback system shown in Fig. 2(a) includes a lead compensator to improve the bandwidth and damping characteristics of the mirror response, and a high-bandwidth "bending filter" to prevent destabilizing feedback of the parasitic structural mode. As shown in Fig. 2(b), the digital system derived by the Tustin transform technique uses a higher sampling rate \( T_s \) versus \( T_c \) for the bending filter to accommodate its high bandwidth. A summary of the algebraic correspondence of the continuous and digital compensator/filter coefficients for this example is presented in Appendix A.

Although the transform method is a convenient means of obtaining a difference equation format for a set of desired control laws, it entails some practical drawbacks. Owing to lags introduced by the approximation, one typically has to retune the digital control laws through simulation experiments to obtain a good emulation of the characteristics of the base continuous-time design. Furthermore, in a manner typical of numerical integration techniques, accuracy of the approximation improves with reduced sample period; hence, use of unnecessarily high sample rates may be encouraged with this technique. These drawbacks notwithstanding, the transform technique has been the basis of a number of successful control designs (which are discussed in the section "Example Multirate Control Systems").

Rattan's Method

Recently, Rattan [18, 19] has developed a technique for multiloop-multirate control design that follows the spirit of simplicity of the transform technique, but yields much more satisfactory results. Rattan's method involves establishing a desired analog control design that meets specifications and an equivalent discrete-domain control structure having unknown compensator coefficients. The unknown coefficients of the discrete-time compensators are then derived to achieve a weighted best-least-squares fit of the phase and gain characteristics (Bode plots) of the digital design to those of the base continuous-time design.

Preliminary results [18, 19] have shown that very good emulation of the base continuous-
time design can be achieved by Rattan's technique. In addition, the technique allows the flexibility to modify the order of the digital control laws, increasing the order to improve the accuracy of the approximation or decreasing it to economize computation.

Widespread use of the Rattan method, as with any new technique, will undoubtedly await the availability of required design software. Also, this method is classically-based and may require further development for use in multi-input/multi-output applications.

Vector Switch Decomposition

Vector switch decomposition is a direct-digital design technique; i.e., design is performed in the discrete-domain as opposed to developing a discrete emulation of a continuous-time design. The basis of the technique is to represent a multirate control structure as a single-rate discrete controller operating at the lowest common sample rate of the multirate controller. This representation is achieved by replacing each of the high rate samples by a low rate sampler flanked by advance and retard functions as depicted in Fig. 3. Once the single-rate representation has been achieved, classical discrete domain “design by analysis” techniques, such as z-plane root locus and w-plane Bode analysis, can be applied to derive an acceptable control law.

The major limitation of the vector-switch approach is the dimensionality growth that it entails. Referring once again to Fig. 3, the function of the advance circuit is to collect the current value of the feedback quantity plus \( \ell - 1 \) future values of the quantity (represented by “advance” transforms), where \( \ell \) is the ratio of the original fast sampler rate to that of the slow sampler. These \( \ell \) quantities are then sampled by the low-rate sampler and “dispatched” one-at-a-time to the next element of the control structure by the retard function. In effect, the single signal crossing the original fast sampler has been replaced by \( \ell \) signals sampled at the low rate; this multiplicative growth of dimensionality can represent a real limitation in application.

Mathematical details of the vector switch decomposition technique — including some useful tables of advance and retard \( z \) transforms — are presented by Whitbeck and Hofmann [8]. Related frequency domain analysis techniques and example applications of the technique are included in [9, 10].

Optimal Control Techniques

In the late 70’s, researchers at the Analytic Sciences Corporation (TASC) developed an approach to multiple sample rate control design based on an optimal control formulation. The net objective of this research was to extend well-established single-rate optimal regulator [20] and Kalman filtering [21] techniques to the multirate case. The design advantages resulting from this technique include:

- Transformation of a base continuous-time design to a multirate structure without approximation.
- Minimal dimensionality growth.
- Guaranteed closed-loop stability.

Transformation from a base continuous-time design to an equivalent multirate design is accomplished by an extension of the method outlined in [20] for the single-rate regulator case, i.e.:

1) A continuous-time optimal regulator is designed to meet continuous-time performance specifications.

2) Discrete-time regulator weighting matrices for the single-rate case are derived from those used to design the continuous-time regulator.

3) Discrete-time periodic weighting matrices used to solve for the multirate gains are constructed from the single-rate matrices. The multirate gains are derived from the periodic steady-state solution of the discrete-time Ricatti equation.

In short, the designer chooses design parameters (weighting matrices) to achieve an acceptable continuous-time design; design

![Diagram](image-url)
parameters for the multirate controller are then derived mathematically from the continuous-time parameters. The mathematics of this transformation are detailed in [13, 14]; as these references indicate, a majority of the computational procedures in this technique are covered by existing single-rate design software packages (such as ORACL, [22]).

The structure of the multirate optimal regulator for the case of two sample rates is shown in Fig. 4. As the block diagram indicates, one control channel, \( u_s \), is updated at a fast rate, \( T_s^{-1} \) samples per second; \( u_s \) is computed at a slower rate, \( (T_r)_{-1} \) samples per second, and is held between computations by a holding circuit. Re-computation of \( u_s \) is accomplished by adding an increment, \( u_k \), to the holding circuit on cycles when \( k = i \cdot \xi \), on all other cycles \( u_k \) is set to zero. Another result of the design procedure is that the slow control is crossfed to the fast control channel. The purpose of the crossfeed is to compensate excitations of the fast modes of the plant caused by \( u_s \) on cycles between slow control updates.

The periodic gains — \( C_{fr}, C_{fs}, \) and \( C_{fl} \) — are obtained by propagating the optimal regulator Riccati equation from infinity backwards to steady state. In the multirate case, the steady-state solution is periodic with period \( \xi \) [13]. The dimensionality of the plant and, hence, the dimension of the Riccati equation, is increased by the number of low rate controls; this dimensionality growth is usually small (an increment of one or two in typical flight control applications) and is generally much smaller than the multiplicative growth entailed by the vector-switch method.

Additional topics covered in [13-16] include multirate estimator design to provide full-state feedback with a limited measurement set, and an algorithmic method for sample rate selection based on a performance/computation tradeoff.

Example Multirate Control Systems

Two currently operational multirate control applications and results of a recent research example are discussed in this section. The intent of this discussion is to highlight the design requirements of these systems that served as drivers for a multirate implementation and briefly to describe the characteristics of the resulting designs.

Space Shuttle Autopilot

The Space Shuttle Orbiter autopilot (specifically, the digital control laws used during reentry and approach and landing) are prime examples of a digital control implementation that must compensate dynamic modes of disparate natural frequencies and accommodate multiple information rates. In addition to providing control and compensation of the rigid body modes of the vehicle, the autopilot includes instrument output filtering to attenuate structural mode response components; the relatively high frequencies of the vehicle structural modes require corresponding high sample rates of the structural filter subroutines.

As a result of the disparity of sampling frequency requirements between structural filtering and rigid body control, a multirate structure was used. This structure was derived by Tustin transform techniques from an analog control design developed on a manned simulator. A sampling frequency high enough to provide effective structural filtering was chosen as the base frequency (25 Hz); lower bandwidth compensators associated with rigid body modes (such as a yaw rate washout filter) were implemented at lower sample rates to reduce computer throughput load.

Although the digital form of the Shuttle control laws was derived by approximation techniques (Tustin transform), the control development process was supported by significant amounts of analysis, simulation, in-
flight simulation and preliminary flight test (i.e., the approach and landing tests of Enterprise during 1978). While the Tustin approximation approach may be considered rudimentary given the power of modern direct-digital design techniques, the Space Shuttle illustrates that successful designs can be derived by approximation techniques when supported by appropriate analysis and evaluation.

**F-18 Digital Fly-By-Wire**

The F-18 digital-fly-by-wire control system is another example of a successful multirate application. The F-18 control laws involve not only filtering and stabilization functions, but also a significant amount of executive and self-test functions. The control laws themselves were derived in analog form on a manned simulator and were then converted to digital form using transformation techniques.

To alleviate onboard computer throughput requirements the F-18 control laws were implemented using a multirate schedule. Sample rates used in the control system are 80, 40, and 10 Hz; typically high frequency sampling is used for instrument filtering, inner-loop control functions (such as stability augmentation), and self testing functions. Lower frequencies are used for gain scheduling and other low-frequency control functions.

The F-18 control system is an example of a system that was derived by classical continuous-time techniques and converted to a digital configuration by transform techniques, with sample rate selection based on the designer’s analysis and judgment. The F-18 control system illustrates the importance of supporting the design process with analysis, simulation (both man-in-the-loop and in-flight), and flight-test evaluation.

**Ride Qualities Optimization Through Multirate Control**

The final design example of this section is an investigation of closed-loop airframe response to atmospheric turbulence as a function of control sample rates. (This study is documented in [14]). The physical model of the airframe and environment investigated is shown in Fig. 5. A mathematical model of a modern fighter aircraft, the F-14, is “flown” through random vertical turbulence. The rms vertical acceleration response (i.e., rms normal g’s) of the airframe was used as the performance measure with multirate closed-loop control of the stabilator and maneuver flaps. Maintaining vertical accelerations at acceptable low-levels is an important control function with respect to preventing pilot fatigue and, possibly, motion sickness.

A constraint was placed on the throughput requirements (computation rate) of the onboard computer: no more than 20 control channel updates (i.e., either stabilator or maneuver flap) could be performed each second. Within the constraint, the designer can tradeoff the sample rates of the two control channels, updating maneuver flap at a lower rate so that a high rate can be used for the stabilator. Using an optimization technique [14], the “best” sample rates (i.e., those that minimize the airframe rms acceleration response) were determined for a number of flight conditions. Results of this analysis indicated that:

- Use of multirate control generally reduced mean-square acceleration response.
- The strength of the optimum (i.e., the performance sensitivity to changes in the control sample rates) varied significantly with flight condition.

Both of these general results indicate that multirate control is an advantageous degree-of-freedom in design, but that the performance payoff varies as a function of the plant dynamics (i.e., flight condition). A practical design procedure for developing a multirate control implementation evolved from these observations. It includes:

![Fig. 6. Airframe response as a function of flap sample rate.](image-url)
Perform an exhaustive survey of optimal sampling policies as a function of flight condition over the vehicle operational envelope.

Determine the subset of flight conditions that exhibit strong optimums.

Choose a single control schedule that has minimal off-optimal performance degradation over the entire flight envelope.

In [14], for example, the optimal sampling schedule for a flight condition of 20,000 ft altitude at an airspeed of 999 ft/sec was stabilator at 13.3 Hz and maneuver flap at 6.67 Hz. As illustrated by Fig. 6, this flight condition showed significant performance degradation for maneuver flap update frequencies lower than the optimum (the trend was towards an unstable solution at $N_{mf} = 0$ as indicated by the dotted portion of the curve). Use of this same sampling schedule at a flight condition of 20,000 ft altitude and an airspeed of 400 ft/sec would not be optimal (optimal for this second flight condition was stabilator at 16 Hz and maneuver flap at 4 Hz) but would result in little degradation of performance from the optimal due to low performance sensitivity. These observations suggest that in practical designs a thorough survey of flight conditions should be performed to identify strong optimums and that the sampling schedules corresponding to these strong optimums should drive the choice of (a single) sampling schedule for the entire vehicle flight envelope.

### Conclusion

Multirate digital control has been and will continue to be an active area of control system research and application. Multirate applications will continue to be motivated by a need to alleviate control computer throughput requirements and to accommodate sensor information available at multiple rates.

Current and future research efforts in multirate control should concentrate on extension of multirate analysis and design techniques to large-scale (high-order) systems and on developing a method of robustness analysis (parameter insensitivity) as a function of control sample rates. Applicable research results in these two areas would significantly extend the range of application, convenience of use and confidence in multirate design techniques.

Although multirate control applications traditionally have been motivated by aerospace systems, needs for improved performance and safety-of-operation should encourage use of multirate techniques in non-aerospace applications. Such applications would include scheduling of automatic sensing and control of process-systems (such as power generating plants), schedule design for manually-monitored and controlled plants, and design of effective displays.

### References


### Appendix A

**Coefficient Values for the System of Fig. 2**

<table>
<thead>
<tr>
<th>ELEMENT</th>
<th>CONTINUOUS</th>
<th>DIGITAL (Tustin)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Lead Compensator</strong></td>
<td>$a, b$</td>
<td>$a^* = \frac{\frac{1}{a_2} - \frac{a_2}{2}}{1 + \frac{a_2}{2}}$</td>
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<tr>
<td></td>
<td></td>
<td>$b^* = \frac{\frac{1}{b_2} - \frac{b_2}{2}}{1 + \frac{b_2}{2}}$</td>
</tr>
<tr>
<td><strong>Bending Filter</strong></td>
<td>$a_n, b_n$</td>
<td>$a_n^* = \frac{-\left[1 - \left(\frac{a_n T}{2}\right)^2\right] - \left(\frac{b_n T}{2}\right)^2}{1 + \frac{a_n T}{2} + \left(\frac{b_n T}{2}\right)^2}$</td>
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<td>$b_n^* = \frac{\left(1 - \frac{a_n T}{2}\right)^2 + \left(\frac{b_n T}{2}\right)^2}{1 + \frac{a_n T}{2} + \left(\frac{b_n T}{2}\right)^2}$</td>
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<tr>
<td></td>
<td>$a_d, b_d$</td>
<td>$a_d^* = \frac{-\left[1 - \left(\frac{a_d T}{2}\right)^2\right] - \left(\frac{b_d T}{2}\right)^2}{1 + \frac{a_d T}{2} + \left(\frac{b_d T}{2}\right)^2}$</td>
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<td>$b_d^* = \frac{\left(1 - \frac{a_d T}{2}\right)^2 + \left(\frac{b_d T}{2}\right)^2}{1 + \frac{a_d T}{2} + \left(\frac{b_d T}{2}\right)^2}$</td>
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<td>$K^* = \frac{1}{1 + \frac{a_d T}{2} + \left(\frac{b_d T}{2}\right)^2}$</td>
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Book Reviews


The older D’Azzo and Houpis texts [1, 2], which have had wide use, have served well as a basis for the “conventional” portion of the current text. “Modern” concepts appear in two ways in this text: first, blended into the flow of the old “conventional” topics where they fit very effectively; and second, as special topics in chapters added to introduce, develop and demonstrate modern techniques.

The four-page preface could serve very adequately as an objective non-critical review of the text. The reviewer recommends that any interested party read this preface for an accurate list of the text contents. Rather than duplicate the preface or table of contents, the reviewer points out the special characters of this text that have led him to use it in his introductory control since the book came into print in 1981.

In Chapter One the reader is introduced to the concept of feedback-control systems starting with those with which even the least-mature engineering student is familiar.

Chapter Two centers on writing the modeling differential equations for a wide variety of systems. It requires the reader to have only the background of integral calculus and minimal experience with matrix algebra. It very gently, but carefully, introduces the “modern” concepts of state-differential equations as a natural flow of events. It takes great care to establish the analogies between electrical, mechanical, thermal, and fluid systems. Motors are introduced as actuators rather than a complex arrangement of windings, valves, etc. The chapter concludes with an introduction to the method of Lagrange Equations, a very powerful tool for coupled systems.

The solutions of the equations written in Chapter Two are treated in Chapter Three. First, the solution of classical linear differential equations is detailed with special emphasis on second order system performance. A well coordinated presentation of solution of state equations concludes the chapter. Good demonstrative examples are provided throughout.

Laplace-transform methods are the stress of Chapter Four. This can very adequately serve as an introduction for those who have had no experience with Laplace transforms, yet at the same time serve as a solid reference to experienced users. Of special interest is the manner in which the authors treat partial fraction expansion, i.e., evaluating residues graphically. This tends to give the reader a good physical feel for what they often see as abstract mathematics. A solid understanding of this method provides the user a powerful tool for specifying system response and how it varies as system singularities are moved around and added or deleted. This is fundamental to efficient design. The chapter concludes with an introduction to Laplace-transform solution of state equations.

Chapter Five establishes the “system” concept, with special emphasis on feedback, block diagrams, signal flow graphs, and multiloop systems. This makes sure the reader understands how complicated things can be, where later chapters deal with much less complex systems demonstrating new concepts. This starts with the classical methods and then moves very smoothly into the “modern” matrix methods. At this point the matrix methods are essentially put under cover until they come out full force in Chapter Twelve.