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A Minimum Realization Method

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Abstract—A new method of obtaining a minimum state space \((A, B, C, D)\) realization of an \(r \times m\) proper rational transfer function matrix, \(P(s)\), is presented. \(D\) is found in the usual manner. The denominator roots are calculated and the \(A\) matrix is formed. An initial estimate of the \(B\) and \(C\) matrices is assigned and a transfer function matrix is calculated from the estimated state space matrices. The \(B\) and \(C\) matrices are adjusted by the algorithm until the computed transfer function is "close enough" to the original transfer function matrix.

Introduction

Implementation of systems in state space form requires significantly fewer integrators and calculations than systems in a transfer function matrix form. This makes the state space form more desirable. Unfortunately, if only the transfer function representation is available, it is not a trivial task to generate a good state space realization. Older methods, Sain [1], Chen [2], provide a minimum realization if the coefficients of the transfer function matrix are not exactly represented. In fact, my interest in this area began when I computed a 2 input, 2 output, 5 state \(P(s)\) from \(A, B, C, D\) matrices, fed it into a program and got a ten-state result. Newer methods, i.e., Foster [6], allow for some variations in the coefficients by specifying an epsilon to define a finite area region around zero as zero. These methods calculate a size for the \(A\) matrix which may not be the minimum. My method requires specification of the size of the \(A\) matrix. If physical insight is available, the size can usually be correctly selected the first time. The method returns an error criterion and a \(P(s)\) is calculated from the computed state space description so a comparison with the given \(P(s)\) is available.

Mathematical Background

A power series representation of a function of a scalar variable \(\lambda\) is

\[
f(\lambda) = \sum_{i=0}^{\infty} a_i \lambda^i
\]

for a region of convergence which is determined by the nature of the function. For example,

\[
\sin(\lambda) = \lambda - \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} - \frac{\lambda^7}{7!} + \ldots
\]

for all real \(\lambda\). The region of convergence is infinite.

\[
(1 - \lambda)^{-1} = 1 + \lambda + \lambda^2 + \lambda^3 + \ldots
\]

Here, the region of convergence is between negative and positive one. The function \(f\) of a square matrix \(A\) is defined [3] as

\[
f(A) = \sum_{i=0}^{\infty} a_i A^i
\]

if the absolute values of all the eigenvalues of \(A\) are smaller than the radius of convergence or the matrix has the property \(A^k = 0\) for some positive integer \(k\). In other words, the terms in the summation either become zero or are finite. Thus

\[
\sin(A) = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \ldots
\]

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for any $A$.

\[(I-A)^{-1} = I + A + A^2 + A^3 \ldots\]

if the eigenvalues of $A$ are between positive and negative one or if $A^t = 0$ for some positive integer $k$.

\[(I-A/s)^{-1} = l/s + A/s^2 + A^2/s^3 + \ldots\]

\[s^{-1}(I-A/s)^{-1} = (sl-A)^{-1}\]

\[= l/s + A/s^2 + A^2/s^3 + \ldots\]

\[(sl-A)^{-1} = l/s \sum_{k=0}^{\infty} s^{-k} A^k\]

Later it will be shown that when equation (1) is used in equation (2) a finite number of terms result. This means the series representation for $P(s)$ will converge.

**Analysis**

Consider an $m$ input, $r$ output, $n$ state transfer function matrix $P(s)$, where $P(s)$ is proper and rational and $n(s)$ is the least common denominator. $n(s)$ is of order $n$ where $n$ is less than or equal to $nn$.

\[P(s) = \frac{M(s)}{n(s)} = C(sl-A)^{-1} B + D\]

where $A$ is a $nn \times nn$ constant coefficient matrix, $B$ is a $nn \times m$ constant coefficient matrix, $C$ is a $r \times nn$ constant coefficient matrix, and $D$ is a $r \times m$ constant coefficient matrix.

\[n(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n s^0\]

\[D = \text{the limit as } s \text{ goes to infinity of } P(s) \text{ or } D = P(s) = \frac{M(\infty)}{n(\infty)}\]

Define

\[M'(s) = M(s) - Dn(s) = C(sl-A)^{-1} Bn(s)\]

Substituting for $(sl-A)^{-1}$ gives

\[M'(s) = 1/s \times C \sum_{k=0}^{\infty} s^{-k} A^k Bn(s)\]

Substituting for $n(s)$ gives

\[M'(s) = 1/s \times C \sum_{k=0}^{\infty} s^{-k} A^k B \sum_{l=0}^{n} s^l a_{n-l}\]

Rearranging the summations and combining terms gives

\[C \left[ \sum_{l=0}^{n} \sum_{k=0}^{\infty} s^{-k-l-1} A^k a_{n-l} \right] B\]

Substitute $k' = -k + l - 1$ and $k = -k' + l - 1$

so that the $A$ matrix and the denominator coefficients have a common term. This gives

\[C \left[ \sum_{l=0}^{n} \sum_{k'=0}^{\infty} s^{k'} A^{-k'+l-1} a_{n-l} \right] B\]

Adjusting the $k'$ summation gives

\[C \left[ \sum_{l=0}^{n} \sum_{k'=l-1}^{\infty} s^{k'} A^{-k'+l-1} a_{n-l} \right] B\]

Breaking this up into positive and negative $k'$ gives

\[C \left[ \sum_{l=0}^{n} \sum_{k'=0}^{\infty} s^{k'} A^{-k'+l-1} a_{n-l} \right] B\]

Note that a zero $l$ gives a negative $k'$ so in the positive $k'$ summation $l$ goes from 1 to $n$ and in the negative $l$ goes from 0 to $n$.

The negative summation equals

\[\sum_{k'=0}^{\infty} s^{k'} A^{-k'-1} \sum_{l=0}^{n} A^l a_{n-l}\]

\[\sum_{l=0}^{n} A^l a_{n-l} = I a_n + A a_{n-1} + A^2 a_{n-2} + \ldots + A^{n-1} a_1 + A^n a_0\]

The denominator of $P(s)$ is the minimal polynomial of the $A$ matrix [5], that is, the roots of the characteristic polynomial contain the roots of the minimal polynomial and the minimal equation is the lowest order equation that the matrix $A$ can satisfy [4]. Since the $A$ matrix satisfies the minimal equation, the above summation equals zero. This means

\[M'(s) = C \left[ \sum_{l=1}^{n} \sum_{k'=0}^{l-1} s^{k'} A^{-k'+l-1} a_{n-l} \right] B\]
Specifying a Unique Solution

For any $n \times n$ real matrix $A$, there exists a real $n \times n$ matrix $E$, with an inverse such that

$$\bar{A} = EAE^{-1}$$

where $\bar{A}$ is a $n \times n$ real block diagonal matrix. All the blocks are in one of three forms

$$[\begin{array}{cc}
-a & 1 \\
0 & -a \\
\vdots & \vdots \\
0 & -a
\end{array}]$$

for real, complex, and $m$-coupled eigenvalues.

A real, nonsingular, block diagonal matrix $W$ exists such that

$$A = W^{-1}A\bar{W}$$

(4)

The blocks of $W$ are chosen to match the three forms of $A$.

$$[\begin{array}{cc}
w_1 & -w_2 \\
w_2 & w_1
\end{array}]$$

(5)

This shows that an infinite number of transformations, $\bar{W}$, exist which will not change the $A$ matrix, but will change the $B$ and $C$ matrices. This means that a unique realization cannot be found from a $P(s)$ if only the $A$ and $D$ matrices are known. To find a unique $A$, $B$, $C$, $D$ realization for a $P(s)$ matrix, it is necessary to specify an $A$ matrix with the proper eigenvalues and one non-zero element in either every column of $C$ or every row of $B$. The remaining elements of $B$ and $C$ may then be obtained by solving the following equation:

$$M(s) - Dn(s) = \sum_{k=0}^{n-1} s^k \sum_{i=k}^{n-1} a_{i,k}CA_{i-1}B$$

(6)

For convenience the $A$ matrix may be chosen with its real roots along the diagonal and its complex roots, $-a \pm jb$, in a $2 \times 2$ real block of the form

$$[\begin{array}{cc}
-a & -b \\
b & a
\end{array}]$$

along the diagonal. The selected $B$ and $C$ matrix elements are set equal to one.

**Example**

$$P(s) = \begin{bmatrix} s+6 & s+3 \\ 4 & s+3 \end{bmatrix}$$

$$D = P(\infty) = \begin{bmatrix} \infty & \infty \\ \infty^2 & 0 \end{bmatrix}$$

$$n(s) = s^2 + 5s + 6$$

$$a_0 = 1 \ a_1 = 5 \ a_2 = 6$$

$$\lambda = -2, -3$$

$A$ is chosen to be

$$\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$
The first row of $C$ is set equal to one. The state space equations are

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} a & b \\ c & d \end{bmatrix} y$$

$$y = \begin{bmatrix} 1 & 1 \\ e & f \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u$$

Since $D = 0$, $M'(s) = M(s)$

$$M'(s) = \begin{bmatrix} s + 6 & s + 3 \\ 4 & s + 3 \end{bmatrix}$$

For this example $n = 2$ so from equation (2)

$$M'(s) = a_6 CB s + a_1 CB + a_0 CAB + 5CB + CAB$$

$$CB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a + c & b + d \\ ae + cf & be + df \end{bmatrix}$$

$$CAB = \begin{bmatrix} -2a - 3c & -2b - 3d \\ -2ae - 3cf & -2be - 3df \end{bmatrix}$$

$$5CB + CAB = \begin{bmatrix} 6 & 3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 3a + 2c & 3b + 2d \\ 3ae + 2cf & 3be + 2df \end{bmatrix}$$

$$a + c = 1 \quad b + d = 1$$

$$3a + 2c = 6 \quad 3b + 2d = 3$$

$$a = 4 \quad b = -1$$

$$c = -3 \quad d = 0$$

$$0 = ae + cf = 4e - 3f$$

$$1 = be + df = e$$

$$e = 1$$

$$f = 4/3$$

$$B = \begin{bmatrix} 4 & 1 \\ -3 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 4/3 \end{bmatrix}$$

**Inexact Coefficients**

For many realizations, the coefficients $P(s)$ may not be known exactly. For these cases

$$R(s) = M'(s) - \sum_{k=0}^{n-1} s^k \sum_{i=k}^{n} a_{l-k}CA^{n-1-l}B$$

where $R(s)$ is a polynomial residual function matrix. Minimizing $R(s)$ does not give good results because there are large variations in the magnitude of the coefficients of $M'(s)$. To correct this each element of $R(s)$ is divided by the corresponding coefficient in $M'(s)$.

$$\frac{m'_{ijk} - s^k \sum_{i=k}^{n} a_{l-k}CA^{n-1-l}B}{m_{ijk}}$$

where $m'_{ijk}$ and $m_{ijk}$ are the coefficients of $s^{k-1}$ in the $i$th element of $R(s)$ and $M'(s)$, respectively. $R(s)$ contains $m \times r \times n$ equations, which are minimized using a program MINPACK1, (Argonne National Laboratories), that employs a least squares algorithm. Generally, if the proper size $A$ matrix has been selected, a close fit will result. Conversely, if the $A$ matrix is too small, the closest fit will be obviously poor.

Two examples are given in the appendix. One is a 2 input, 2 output, 2 state $(2 \times 2 \times 2)$ example, the other is a $6 \times 4 \times 4$ transfer function matrix for which an excellent $A, B, C, D$ match was found. The method has also been tested in 19 other examples ranging from a $2 \times 2 \times 2$ to a $6 \times 4 \times 4$ with equal success.

**Conclusions**

In this paper a new method of obtaining minimum realizations has been introduced. The method has several important features

- Block Diagonal Nature of $A$ matrix
- Least Squares Fit
- Exact Preservation of Poles
- Practical Computational Technique

It is our hope that practicing control engineers will find this a useful method since it always returns an answer.

**Appendix**

The first example is a computer run of the problem worked out in the text.

This example shows the realization of a 6 state, 4 input, 4 output plant. The plant is a reduced order model of a F100 at $PLA = 36$ degrees.

Figure 1a shows the original plant, and Figure 1b the transfer function matrix calculated from the state space representation. The next figure shows the $A, B, C, D$ matrices. The last figures compare the open loop responses of the given plant (Figure 3) and the realized plant (Figure 4). Figure 5 shows the error between the two transient responses.

*Note: $(\tau)$ represents a response of $\tau_s + 1$. $(\tau_1, \tau_2)$ represents a response of $\tau_1 s^2 + \tau_2 s + 1$. 

September 1981
THE FOLLOWING 13 ELEMENTS OF THE C MATRIX HAVE BEEN SET EQUAL TO ONE, 1, 2
THE FLAGS ARE 1, 1, 2, 0, 3, 0, 4, 1, 5, 0, 6, 0, 7, 1, 8, 2, 9, 0.

THE ORIGINAL TRANSFER FUNCTION MATRIX

THE GAINS NUMERATOR MATRIX IS

\[
\begin{align*}
\text{MATRIX ROW 1} & : \\
\text{DEG 0} & : 1.00000 \\
\text{DEG 1} & : 166667 \\
\text{DEG 2} & : 0 \\
\text{MATRIX ROW 2} & : \\
\text{DEG 0} & : .666667 \\
\text{DEG 1} & : 0 \\
\text{DEG 2} & : 0 \\
\text{GAINS TRANSFER FUNCTION DENOMINATOR} & : s^{**1} + 5s^{**2} + 6 \\
\end{align*}
\]

THE CSPM NUMERATION MATRIX IS

\[
\begin{align*}
\text{MATRIX ROW 1} & : \\
\text{DEG 0} & : 6.00000 \\
\text{DEG 1} & : 1.00000 \\
\text{DEG 2} & : 0 \\
\text{MATRIX ROW 2} & : \\
\text{DEG 0} & : 4.00000 \\
\text{DEG 1} & : 0 \\
\text{DEG 2} & : 0 \\
\text{CSPM TRANSFER FUNCTION DENOMINATOR} & : s^{**1} + 5s^{**2} + 6 \\
\end{align*}
\]

THE DENOMINATOR ROOTS ARE

\[
\begin{align*}
\text{REAL} & : -2.00000 \\
\text{IMAG} & : -3.00000 \\
\text{S**1} & : 0 \\
\text{S**2} & : 0 \\
\end{align*}
\]

THE FINAL SOLUTION follows

THE SUM OF THE SQUARES OF THE RESIDUALS IS .2484450x30

\[
\begin{align*}
\text{MATRX-A} & : \\
\text{ROW COLUMNS 1} & : 1 \\
\text{1} & : -2.00000 \\
\text{2} & : -3.00000 \\
\text{MATRX-H} & : \\
\text{ROW COLUMNS 1} & : 2 \\
\text{1} & : 4.00000 \\
\text{2} & : 1.00000 \\
\text{MATRX-C} & : \\
\text{ROW COLUMNS 1} & : 2 \\
\text{1} & : 1.00000 \\
\text{2} & : 1.33333 \\
\text{MATRX-D} & : \\
\text{ROW COLUMNS 1} & : 2 \\
\text{1} & : 0 \\
\text{2} & : 0 \\
\end{align*}
\]

THE TRANSFER FUNCTION MATRIX CONSTRUCTED FROM THE ABOVE A, B, C, D IS

THE GAINS NUMERATOR MATRIX IS

\[
\begin{align*}
\text{MATRIX ROW 1} & : \\
\text{DEG 0} & : 1.00000 \\
\text{DEG 1} & : 166667 \\
\text{DEG 2} & : 0 \\
\text{MATRIX ROW 2} & : \\
\text{DEG 0} & : .666667 \\
\text{DEG 1} & : .740149x15 \\
\text{DEG 2} & : 0 \\
\text{GAINS TRANSFER FUNCTION DENOMINATOR} & : s^{**1} + 5s^{**2} + 6 \\
\end{align*}
\]

THE CSPM NUMERATION MATRIX IS

\[
\begin{align*}
\text{MATRIX ROW 1} & : \\
\text{DEG 0} & : 6.00000 \\
\text{DEG 1} & : 1.00000 \\
\text{DEG 2} & : 0 \\
\text{MATRIX ROW 2} & : \\
\text{DEG 0} & : 4.00000 \\
\text{DEG 1} & : .444089x15 \\
\text{DEG 2} & : 0 \\
\text{CSPM TRANSFER FUNCTION DENOMINATOR} & : s^{**1} + 5s^{**2} + 6 \\
\end{align*}
\]

EXAM PLE 1.

\[
P(s) = \begin{bmatrix} s+6 & s+3 \\ 4 & s+3 \end{bmatrix}
\]

\[
\begin{align*}
P(s) & = \frac{s+6}{s^2 + 5s + 6} \\
\end{align*}
\]

\[
P(s) = \frac{s+6}{s^2 + 5s + 6}
\]

Derived Plant

\[
P(s) = \begin{bmatrix} s+6 & s+3 \\ 10^{-15} & s+3 \end{bmatrix}
\]

\[
\begin{align*}
P(s) & = \frac{s+6}{s^2 + 5s + 6} \\
\end{align*}
\]
Figure 1a

\[
\begin{array}{cccccc}
0.81(1.4)(.56)(.25)(.11)(.002) & 770(1.5)(.57)(.38)(.034)(.014) & -15(1.5)(.58)(.39)(.097)(.02) & -15(1.5)(.5)(.19)(-.19)(.016) \\
0.58(1.4)(.59)(.046)(.44)(.021) & -30(1.5)(-3.1)(.51)(.655)(.02) & 0.67 (-3.9)(1.5)(.5)(.052)(.021) & -26(1.5)(.5)(.33)(.3)(.02) \\
0.003(1.4)(.56)(.24)(-.15)(.023) & -3.5(1.5)(.52)(.22)(.72)(.02) & -.0006(-17.1)(1.2, 2.2)(.5)(.02) & -.03(1.7)(-.96)(.51)(.12)(.02) \\
0.11(1.5)(.47)(.26)(.10)(.49) & 1.8(-5.5)(1.5)(.47)(-.19)(.11) & -14(1.5)(-1.4)(.47)(-.14)(-.12) & 3.9(1.5)(.47)(.17)(.032)(.26) \\
\end{array}
\]

Figure 1b

\[
\begin{array}{cccccc}
0.81(1.3)(.66)(.24)(.11)(.002) & 770(1.4)(.61)(.37)(.034)(.014) & -15(1.4)(.6)(.38)(.098)(.02) & -15(1.5)(.49)(.19)(-.19)(.016) \\
0.58(1.4)(.6)(.049)(.44)(.021) & -30(1.5)(-3.1)(.51)(.665)(.02) & 0.67 (-3.9)(1.5)(.5)(.051)(.021) & -26(1.5)(.5)(.33)(.3)(.02) \\
0.003(1.3)(.59)(.24)(-.15)(.024) & -3.5(1.5)(.51)(.22)(.72)(.02) & -.0006(-17.1)(1.2, 2.2)(.49)(.021) & -.03(1.7)(-.96)(.5)(.13)(.019) \\
0.11(1.5)(.47)(.29)(.096)(.46) & 1.8(-5.6)(1.6)(.45)(-.19)(.11) & -13(1.5)(-1.4)(.45)(-.14)(-.12) & 3.9(1.5)(.47)(.18)(.031)(.25) \\
\end{array}
\]

STATE SPACE REALIZATION

\[
A = \begin{bmatrix}
-0.685 & 0 & 0 & 0 & 0 & 0 \\
0 & -1.85 & 0 & 0 & 0 & 0 \\
0 & 0 & -3.87 & 0 & 0 & 0 \\
0 & 0 & 0 & -2.95 & 2.51 & 0 \\
0 & 0 & 0 & -2.51 & -2.85 & 0 \\
0 & 0 & 0 & 0 & 0 & -49
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-0.0201 & -0.734 & 0.308 & -0.319 \\
-0.804 & -72.5 & 1.65 & -7.04 \\
0.186 & -3060 & 45.6 & -46.5 \\
-1.31 & -871 & -6.22 & 87.1 \\
2.41 & 4510 & -76.9 & -3.63 \\
-1.06 & -78.1 & 1.28 & -6.64
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
2.76 & 1.4 & 1.16 & 0.23 & 0.762 & -0.0755 \\
-0.0177 & 0.00299 & 0.0142 & 0.0103 & 0.00175 & 0.00103 \\
0.146 & -0.199 & -0.0963 & -0.0139 & -0.0847 & -7.46
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
References


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